Causal RG equation for Quantum Einstein Gravity

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14.03.2011

arXiv:1102.5012v1 [hep-th]

with Elisa Manrique and Frank Saueressig

Outline

Motivation

Classical GR reaches its limits close to space-time singularities

- Black Holes
- Big Bang

Solution probably lies within a theory of Quantum Gravity

different approaches to a theory of QG

- String Theory
- Loop Quantum Gravity
- Causal Dynamical Triangulations
- Horava Gravity
- **Asymptotic Safety**
- ...

} which one is correct?

lack of experimental data

$\Rightarrow$ nobody helps us to decide which is the best approach $\Rightarrow$ what to do?

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start crying and wait for data

OR

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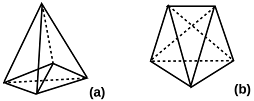
start crying and wait for data

OR

compare different approaches

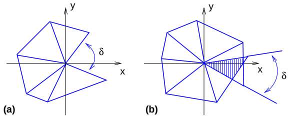
Causal Dynamical Triangulations (arXiv:1004.0352v1 [hep-th])

- discretization of gravitational path integral $\int \mathcal{D}g_{\mu\nu} e^{iS_{\text{grav}}}$ by summing over piecewise flat geometries
- modeling space-time geometries by gluing together simplices (higher dimensional generalizations of triangles)



(arXiv:hep-th/0105267v1)

- curvature introduced via deficit angles
  pos./neg. deficit angles correspond to pos./neg. curved space



(arXiv:hep-th/0509010v3)

- causal structure (equal-sided instead of equilateral triangles)

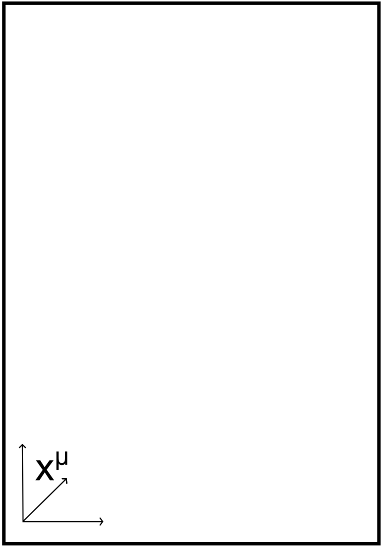
Horava Gravity (arXiv:0901.3775v2 [hep-th])

- different scaling of space and time
  $x \to bx, \quad t \to b^z t$
- Lorentz invariance is broken in UV but reestablished in IR
- foliation-preserving diffeomorphism invariance
  infinitesimal generators: $\delta x^i = \zeta^i(t, \mathbf{x}), \quad \delta t = f(t)$
- Horava-like action:

$$S_H = \frac{1}{16\pi G_{\mathrm{N}}} \int d^3x dt N \sqrt{g} \left[K_{ij} K^{ij} - \lambda K^2 - \gamma R^{(3)} + 2\Lambda + V(g_{ij}) \right]$$

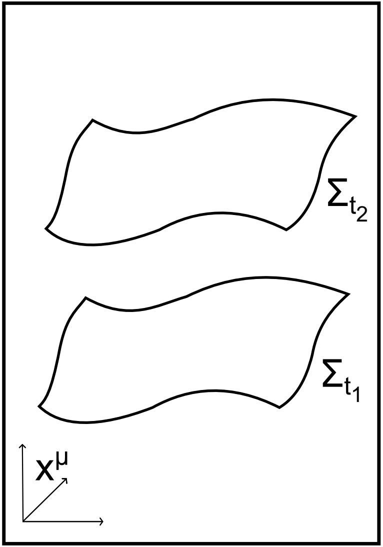
- maybe connection to CDT due to global time foliation
  (arXiv:1002.3298v2 [hep-th])

ADM decomposition



- time function ... $t(x^\mu)$
- $\partial_\mu t$ ... future-dir. timelike VF

ADM decomposition



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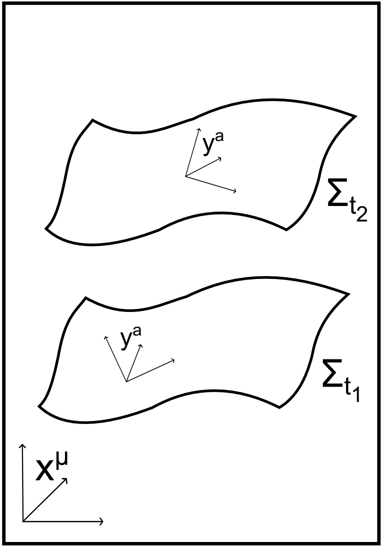
- Σ_{t_i} ... spatial slices

ADM decomposition



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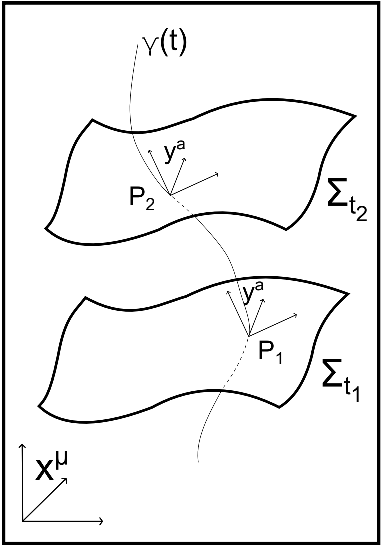

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- parametrized curve $\gamma(t)$
- $y^a(P_1) = y^a(P_2)$

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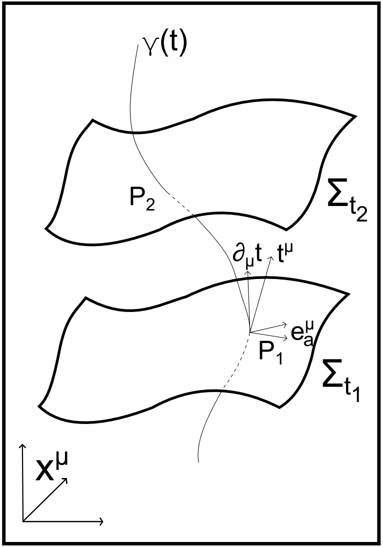

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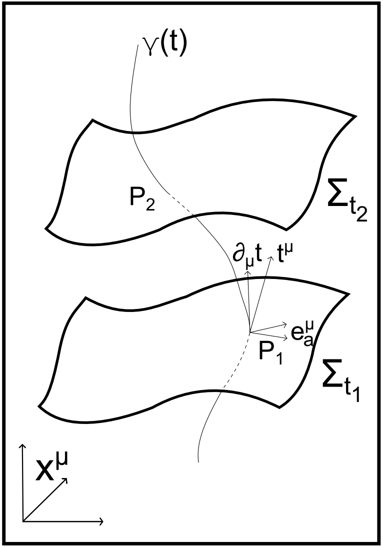

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- $t^\mu = \left(\frac{\partial \gamma^\mu}{\partial t}\right)_{y^a}, \quad e^\mu_a = \left(\frac{\partial \gamma^\mu}{\partial y^a}\right)_t$
- $t^\mu = N^2 \partial^\mu t + N^a e^\mu_a$
- lapse function ... N
- shift vector ... N^a

ADM decomposition

change of coordinates: $x^\mu \leftrightarrow (\tau, y^a)$

old coordinates: $ds^2 = \gamma_{\mu\nu} dx^\mu dx^\nu$

new coordinates: $ds^2 = N^2 d\tau^2 + \sigma_{ij} \left(dy^i + N^i d\tau\right) \left(dy^j + N^j d\tau\right)$

with induced spatial metric: $\sigma_{ij} = \gamma_{\mu\nu} e_i^\mu e_j^\nu$

$$\gamma_{\mu\nu} = \begin{pmatrix} N^2 + N_i N^i & N_j \\ N_i & \sigma_{ij} \end{pmatrix}$$

ADM decomposition

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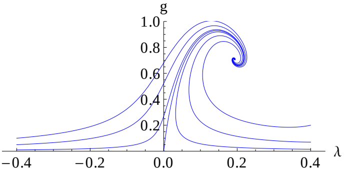
Causal RG equation

Starting point: Einstein Hilbert action

$$S_{\mathrm{EH}} = \frac{1}{16\pi G_{\mathrm{N}}} \int d^D x \sqrt{\gamma}\,(-R + 2\Lambda)$$

- G_{N} ... Newton constant
- D ... space-time dimension $(D = d + 1)$
- γ ... metric
- R ... curvature scalar of space-time
- Λ ... cosmological constant



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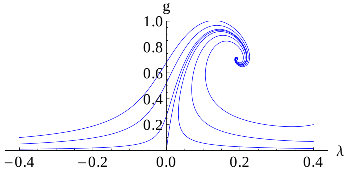

$$S_{\mathrm{ADM}} = \frac{\sqrt{\epsilon}}{16\pi G_{\mathrm{N}}} \int d\tau \int d^d x N \sqrt{\sigma} \left\{ \epsilon^{-1} K_{ij} \left[\sigma^{ik}\sigma^{jl} - \sigma^{ij}\sigma^{kl} \right] K_{kl} - R + 2\Lambda \right\}$$

gauge fixing

background field method

first try: $\gamma_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad \Delta S_k \propto h_{\mu\nu} R_k^{\mu\nu\rho\sigma} h_{\rho\sigma}$
but split nonlinear in $N, N^i, \sigma_{ij} \quad \Rightarrow \quad$ cutoff non-quadratic in N, N^i, σ_{ij}

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but split nonlinear in N, N^i, σ_{ij} $\Rightarrow$ cutoff non-quadratic in N, N^i, σ_{ij}

second try: special background $\bar{g}_{\mu\nu} = \begin{pmatrix} \epsilon & 0 \\ 0 & \bar{\sigma}_{ij} \end{pmatrix} \quad \leftrightarrow \quad \bar{N} = 1, \bar{N}^i = 0$

$\Rightarrow$ gauge freedom can be fixed with $N = 0 = N^i$ (arXiv:hep-th/0103186v2)

$\Rightarrow$ purely spatial fluctuations h_{ij}

ghost action: $S_{\text{gh}} = \sqrt{\epsilon} \int d\tau \int d^d x \sqrt{\bar{\sigma}} \left\{ \bar{C} \partial_\tau C + \bar{C}_i \partial_\tau C^i \right\}$

Fourier decomposition in time direction

time direction $\leftrightarrow$ circle S^1 with interval length T

$$\phi(\tau, x) = \sum_{n=-\infty}^{\infty} \phi_n(x) e^{2\pi i n\tau/T} \qquad \Rightarrow \qquad \phi_n(x) = \frac{1}{T}\int_0^T d\tau \phi(\tau, x) e^{-2\pi i n\tau/T}$$

$$\int_0^T d\tau\, \phi(x,\tau)^2 = T \sum_{n=-\infty}^{\infty} \phi_n(x)\phi_n^*(x)\,,$$

$$\int_0^T d\tau\, \phi(x,\tau)\partial_\tau^2\phi(x,\tau) = -T \sum_{n=-\infty}^{\infty} \phi_n(x)\left(\frac{2\pi n}{T}\right)^2 \phi_n^*(x)$$

regulator and simplifications

regulator: purely spatial $\mathcal{R}_k(\Delta)$ with $\Delta = -\bar{\sigma}^{ij}\bar{D}_i\bar{D}_j$

$\mathcal{R}_k(\Delta)$ is given by implementing: $\Delta \to \Delta + R_k$

for simplicity optimized cutoff: $R_k = k^2\left(1 - \frac{\Delta}{k^2}\right)\theta\left(1 - \frac{\Delta}{k^2}\right)$

further simplification:

transverse-traceless decomposition $h_{ij}(x) \mapsto \left\{h_{ij}^{\mathrm{T}}(x), \xi_i^{\mathrm{T}}(x), \sigma(x), h(x)\right\}$
$+$ field redefinition

$\Rightarrow$ no Jacobians

analytic structure

$$k\partial_k \Gamma_k = \frac{1}{2}\mathrm{STr}\left[\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1} k\partial_k \mathcal{R}_k\right] = T^{\mathrm{TT}} + T^0$$

analytic structure

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$$T^{\mathrm{TT}} = \frac{\sqrt{\epsilon}k^d d_{2\mathrm{T}}}{(4\pi)^{d/2}} \sum_n \int d^d x \sqrt{\bar{\sigma}} \Big[q^{1,0}_{d/2}(w_{2\mathrm{T}}) + \bar{R}k^{-2}\left(\tfrac{1}{6} q^{1,0}_{d/2-1}(w_{2\mathrm{T}}) - \tfrac{d^2-3d+4}{d(d-1)} q^{2,0}_{d/2}(w_{2\mathrm{T}})\right)\Big]$$

$$T^0 = \frac{\sqrt{\epsilon}k^d}{(4\pi)^{d/2}} \sum_n \int d^d x \sqrt{\bar{\sigma}} \Big[q^{1,0}_{d/2}(w_0) + \tfrac{1}{6}\bar{R}k^{-2} q^{1,0}_{d/2-1}(w_0) + \tfrac{d-2}{2d\lambda_k}\bar{R}k^{-2}\Big\{ q^{2,-1}_{d/2}(w_0) - q^{1,0}_{d/2}(w_0) + \left(\tfrac{3}{2\epsilon} m^2 n^2 - \tfrac{4(d-3)}{d-2}\lambda_k\right) q^{2,0}_{d/2}(w_0)\Big\}\Big]$$

analytic structure

$$\sum_n q^{1,0}_{d/2}(w_{2\mathrm{T}}) \propto \sum_n \frac{1}{1 + \frac{1}{2\epsilon} m^2 n^2 - 2\lambda_k}$$

Kaluza-Klein mass: $m = \frac{2\pi}{Tk}$

carry out sums analytically:

$$\sum_n \frac{1}{n^2 + x^2} = \frac{\pi}{x\, tanh(\pi x)}, \quad x^2 > 0 \quad \text{(hyperbolic functions)}$$

$$\sum_n \frac{1}{n^2 + x^2} = \frac{\pi}{i\, x\, tan(\pi x)}, \quad x^2 < 0 \quad \text{(trigonometric functions)}$$

flow equation

$$k\partial_k g_k = \beta_g(g,\lambda;m), \qquad k\partial_k \lambda_k = \beta_\lambda(g,\lambda;m)$$

$$m = \frac{2\pi}{Tk}$$

analytic structure depends on signature and on λ

ϵ	$\lambda < \lambda^{(1)} < 0$	$\lambda^{(1)} < \lambda < \lambda^{(2)} = 1/2$	$\lambda^{(2)} < \lambda$
$+1$	hyperbolic	mixture	trigonometric
-1	trigonometric	mixture	hyperbolic

Gaußian fixed point scenario

$$m = \frac{2\pi}{Tk} \qquad T = const$$

$$\Rightarrow \quad k\partial_k m_k = -m_k \quad \Rightarrow \quad m_k^* = 0$$

in this limit all trigonometric functions diverge

ϵ	$\lambda < \lambda^{(1)} < 0$	$\lambda^{(1)} < \lambda < \lambda^{(2)} = 1/2$	$\lambda^{(2)} < \lambda$
$+1$	hyperbolic	~~mixture~~	~~trigonometric~~
-1	~~trigonometric~~	~~mixture~~	hyperbolic

$\Rightarrow$ NGFP only for $g_* < 0$ in Euclidean signature!

floating point scenario

$$m = const. \quad \Rightarrow \quad T \propto \frac{1}{k}$$

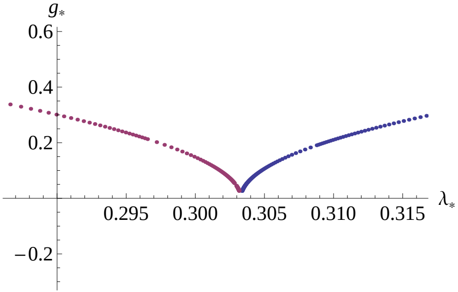
if $m > \sqrt{5/2}$ trigonometric terms stay finite for $\lambda^{(1)} < \lambda < \lambda^{(2)}$

Example: $m = 2\pi \quad \Rightarrow \quad T = \frac{1}{k}$

Non-Gaußian Fixed Point in Euclidean **and** Lorentzian signature

ϵ	g_*	λ_*	$g_*\lambda_*$	$\theta_{1,2}$
$+1$	0.19	0.31	0.059	$1.07 \pm 3.31i$
-1	0.21	0.30	0.063	$0.94 \pm 3.10i$

Fixed Points depending on m

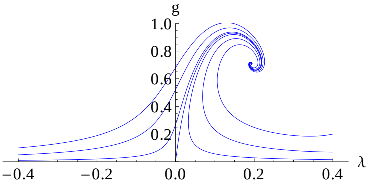


Fixed-Point values depending on m

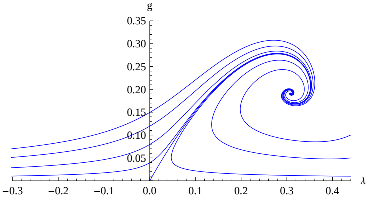
Euclidean FP ... blue Lorentzian FP ... red

both FPs converge for increasing Kaluza-Klein mass

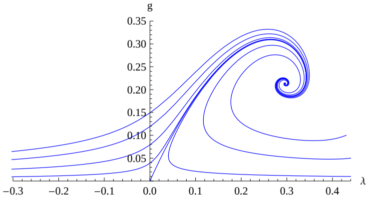
Trajectories



covariant calculation



Euclidean



Lorentzian

Conclusion

- FP for Euclidean and Lorentzian signature
- characteristics are similar
- also similar to covariant formulation
- time circle collapses toward UV
- signature does NOT matter in UV
- formulation prepares ground for comparison to other theories

Thank you for your attention!

Questions?