Menu

Renormalisation Group improvement

RG-improved cosmology

Conclusions

Renormalisation Group improvement of early times cosmology

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Asymptotic Safety Seminars

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À la carte

Entrée

few thoughts about the concept of RG improvement

Plat principal

RG improvement of scalar field inflation

Dessert

ripe conclusions and juicy prespectives

Renormalisation Group improvement

RG is a way to encode quantum corrections *in* coupling constants

so that tree-level description can be sufficient

application to cosmological (and astrophysical) settings

$\Downarrow \quad \Downarrow \quad \Downarrow$

inclusion of QG effects in (high energy) processes

Few useful concepts about RG improvement *à la* Wilson

wise choice of scale k $\Rightarrow$ tree-level description

BUT

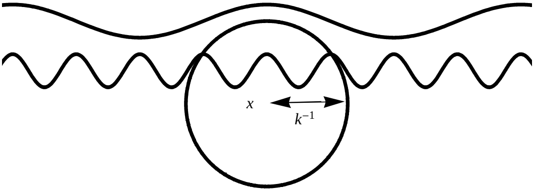
in general, momentum scales are not conserved over a curved manifold

$\Longrightarrow$ we are forced to assign x-dependence to k

$\Longrightarrow$ coupling constants acquire x-dependence too

Another concept worth being recalled:

EFT at scale k $\Rightarrow$ integration of modes $p > k$



effects inside radius are already encoded in the effective action
(no need to consider them when dealing with fluctuations)

fluctuations of couplings outside radius are not encoded
$\delta g \propto \delta k$ *must* be taken into account!

Several ways to implement RG improvement:

- *EXTENDED* IMPROVEMENT

consider couplings as "external fields" in the action

$$\Gamma_{EH} = -\frac{1}{16\pi G(x)} \int d^4x \sqrt{-g}\, R$$

when varying the action one gets additional terms

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - G\left(\nabla_\mu \nabla_n - g_{\mu\nu} \nabla^2\right) G^{-1} = 8\pi G T_{\mu\nu}$$

- *RESTRICTED* IMPROVEMENT

neighbourhood $\mathcal{I}$ of event $A \quad \Rightarrow \quad k_A$, action Γ_A

EOM in $\mathcal{I}$ read $\quad G_{\mu\nu} = 8\pi G_A T^A_{\mu\nu}$

iterating for *each* event x gives

$$G_{\mu\nu} = 8\pi G_k T^k_{\mu\nu}$$

with the additional x-dependence carried by $k(x)$

Renormalisation Group improved cosmology

Gravity (minimally) coupled to scalar field ϕ

$$\Gamma_k[g,\phi] = \int d^4x\sqrt{-g}\left[-\frac{R}{16\pi G_k} + \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi + V_k(\phi)\right]$$

and the potential is polynomial

$$V_k(\phi) = \sum \lambda_{2i}\,\phi^{2i}$$

Equations of motion

$$\begin{aligned} G_{\mu\nu} &= 8\pi G_k T^k_{\mu\nu} \\ \Box\phi &= V'_k(\phi) \end{aligned}$$

where

$$T^k_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla_\rho\phi\nabla^\rho\phi - V_k(\phi)$$

Last ingredient is the definition of function $k(x)$ defined implicitly using conservation laws

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Diffeomorphism invariance of *each* action gives conservation laws

$$\nabla_\mu G^{\mu\nu} = 0$$

$$\nabla_\mu \left. T^{\mu\nu}\right|_\lambda = 0$$

while EOM give the overall conservation

$$\nabla_\mu \left(G T^{\mu\nu}\right) = 0 \qquad \Rightarrow \qquad \nabla_\mu G T^{\mu\nu} - G \left. \nabla^\nu V(\phi)\right|_\phi = 0$$

New constraint $\Rightarrow$ equation for $k(x)$

k-constraint

$$(\nabla_\mu \ln k) T^{\mu\nu} \eta_{\rm RG} = (\nabla^\nu \ln k) V \nu_{\rm RG}$$

where

$$\eta_{\rm RG} = \frac{\partial \ln G}{\partial \ln k} = \frac{\beta_{\tilde{G}}}{\tilde{G}} - 2$$

$$\nu_{\rm RG} = \frac{\partial \ln V}{\partial \ln k} = \frac{1}{V} \sum_i \left(\beta_{\tilde{\lambda}_i} + (4-i)\tilde{\lambda}_i \right) k^{4-i} \phi^i$$

being

$$\nu_{\rm RG} \equiv \nu_{\rm RG}(\tilde{\phi})$$

$$G(k) = k^{-2}\tilde{G}(k) \quad ; \qquad \lambda_i(k) = k^{4-i}\tilde{\lambda}_i(k) \quad ; \qquad \tilde{\phi} = k^{-1}\phi$$

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Fixed point regime

imposing FRW symmetry

$$
\begin{aligned}
H^2 &= \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V\right) \\
\dot{H} &= -4\pi G \dot{\phi}^2 \\
\ddot{\phi} &= -3H\dot{\phi} - V' \\
\dot{\phi}^2 &= -2\left(1 + \frac{\nu_{\mathrm{RG}}}{\eta_{\mathrm{RG}}}\right) V
\end{aligned}
$$

UV fixed point $\Rightarrow$ $\tilde{G}(k) \simeq \tilde{G}^* \, , \; \tilde{\lambda}_i(k) \simeq \tilde{\lambda}_i^*$

$$
\frac{\dot{H}}{H^2} = -\frac{1}{\alpha} \qquad \text{where} \qquad \alpha = \frac{1}{3\left(1 + \frac{\eta_{\mathrm{RG}}}{\nu_{\mathrm{RG}}}\right)}
$$

Ansatz $H = \alpha/t \quad , \quad \phi = \varphi/t \quad , \quad k = \chi/t$

$$\alpha^2 = \frac{8\pi\tilde{G}}{3}\left(\frac{1}{2}\tilde{\phi}^2 + \chi^2\tilde{V}\right)$$

$$\varphi^2 = -\frac{\tilde{\phi}^4}{2\left(1+\frac{\nu_{\rm RG}}{\eta_{\rm RG}}\right)\tilde{V}}$$

$$\chi^2 = -\frac{\tilde{\phi}^2}{2\left(1+\frac{\nu_{\rm RG}}{\eta_{\rm RG}}\right)\tilde{V}}$$

where $\tilde{V} = k^{-4}V$ is a function of $\tilde{\phi}$ only

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MONOMIAL POTENTIAL

$$V(\phi) = \lambda_n \, \phi^n$$

$$\alpha = \frac{4-n}{3\,(2-n)}$$

$$\chi^2 = \frac{1}{(2-n)\tilde{\lambda}_n} \left(\frac{4-n}{12(2-n)\pi \tilde{G}} \right)^{\frac{2-n}{2}}$$

$$\varphi^2 = \frac{1}{(2-n)\tilde{\lambda}_n} \left(\frac{4-n}{12(2-n)\pi \tilde{G}} \right)^{\frac{4-n}{2}}$$

$\nexists\, n$ such that $\alpha > 1$

<u>no viable inflationary solution</u>

REMARK:

- α does not depend on the FP values of couplings
- φ only depends on the "dimensionless" combination $\tilde{\lambda}_n \, \tilde{G}^{\frac{2-n}{2}}$
- χ depends on a "dimensionful" combination

general feature of solutions, maybe because H and ϕ are physical while k is only a RG parameter?

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TRINOMIAL POTENTIAL

$$V(\phi) = \lambda_0 + \lambda_2\,\phi^2 + \lambda_4\,\phi^4$$

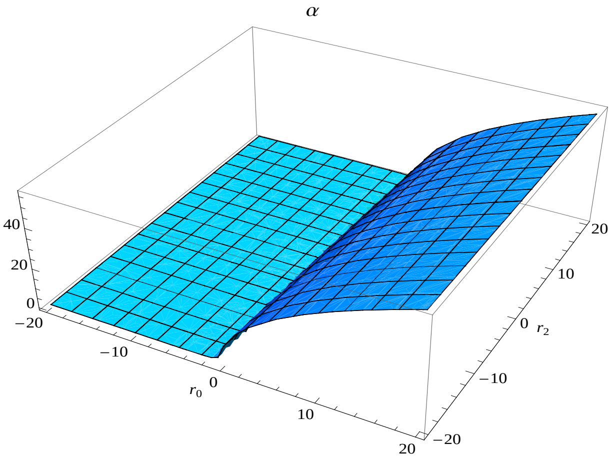
$$\alpha = \frac{2\tilde{\lambda}_0 + \tilde{\lambda}_2\tilde{\phi}^2}{3(\tilde{\lambda}_0 - \tilde{\lambda}_4\tilde{\phi}^4)}\,; \quad \varphi^2 = \frac{\tilde{\phi}^4}{2(\tilde{\lambda}_0 - \tilde{\lambda}_4\tilde{\phi}^4)}\,; \quad \chi^2 = \frac{\tilde{\phi}^2}{2(\tilde{\lambda}_0 - \tilde{\lambda}_4\tilde{\phi}^4)}$$

and $\tilde{\phi}$ is given by

$$\tilde{\lambda}_0 - \tilde{\lambda}_4\tilde{\phi}^4 = \frac{1}{12\pi\tilde{G}}\left(\frac{2\tilde{\lambda}_0}{\tilde{\phi}^2} + \tilde{\lambda}_2\right)$$

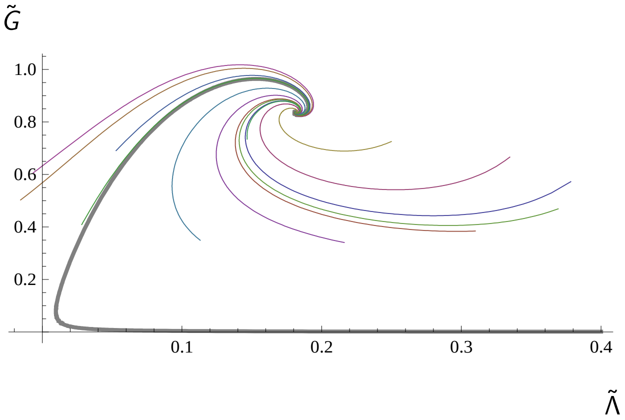
Redefining FP values of couplings into

$$r_0 = \tilde{\lambda}_0/\tilde{\lambda}_4 \text{ and } r_2 = \tilde{\lambda}_2/\tilde{\lambda}_4$$



as computations generally give $\tilde{G} = \mathcal{O}(1)$

Quasi-classical regime



long-lasting phase of almost classical evolution, trajectory close to Gaussian fixed point vicinity of Gaussian FP $\Rightarrow$ *small* couplings

linearised β functions:

$$\beta_{\tilde{\Lambda}} = \frac{3\,\tilde{G}}{4\pi} - 2\,\tilde{\Lambda} \qquad \beta_{\tilde{\lambda}_2} = -2\,\tilde{\lambda}_2 - \frac{3\,\tilde{\lambda}_4}{8\pi^2}$$

$$\beta_{\tilde{G}} = 2\,\tilde{G} \qquad \beta_{\tilde{\lambda}_4} = 0$$

[Narain & Percacci '10]

linearised flux can be integrated analitically

$$\Lambda(k) = \bar{\Lambda} + \frac{3}{16\pi}\,\bar{G}\,k^4 \qquad \lambda_2(k) = \bar{\lambda}_2 - \frac{3}{16\pi^2}\,\bar{\lambda}_4\,k^2$$

$$G(k) = \bar{G} \qquad \lambda_4(k) = \bar{\lambda}_4$$

GFP means vanishing $\eta_{\rm RG}$ $\Rightarrow$ k-constraint implies $\nu_{\rm RG} = 0$

for trinomial potential

$$k(t) = 2\sqrt{\bar\lambda_4}\,\phi(t)$$

field equation can be written in a closed form

$$\ddot\phi + 2\sqrt{6\pi\bar G\left(\frac{1}{2}\dot\phi^2 + \frac{\bar\Lambda}{8\pi\bar G} + \bar\lambda_2\,\phi^2 + \left(1 - \frac{3\bar\lambda_4}{8\pi^2}\right)\bar\lambda_4\,\phi^4\right)}\,\dot\phi = -2\left(\bar\lambda_2 + 2\left(1 - \frac{3\bar\lambda_4}{8\pi^2}\right)\bar\lambda_4\,\phi^2\right)\phi$$

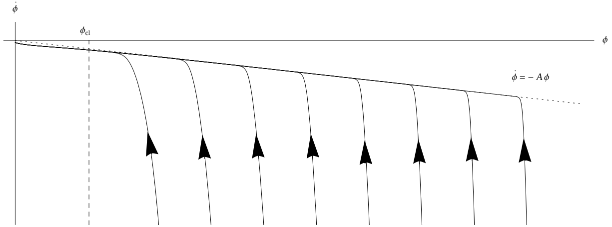
and studied in the phase space

Kinetic-dominated phase:

$$\dot{\phi}(\phi) \simeq \Phi\, e^{-\sqrt{12\pi \bar{G}}\,\phi}$$

Potential-dominated phase:

$$\dot{\phi}(\phi) \simeq -\sqrt{\frac{2}{3\pi \bar{G}} \left(1 - \frac{3\,\bar{\lambda}_4}{8\pi^2}\right) \bar{\lambda}_4}\, \phi$$



Attractor solution $\dot{\phi}(\phi) = -A\phi$

“classical threshold” $\phi_{\rm cl}$

$$\begin{aligned}\Lambda(\phi) &= \bar\Lambda + \frac{3}{\pi}\,\bar G\,\bar\lambda_4^2\,\phi^4 & \lambda_2(\phi) &= \bar\lambda_2 - \frac{3}{4\pi^2}\,\bar\lambda_4^2\,\phi^2 \\ G(\phi) &= \bar G & \lambda_4(\phi) &= \bar\lambda_4\end{aligned}$$

$$\phi_{\rm cl} = \min\{\phi_0, \phi_2\}$$

$$\phi_0 = \left(\frac{\pi}{3}\,\frac{\bar\Lambda}{\bar G\,\bar\lambda_4^2}\right)^{1/4} \quad ; \quad \phi_2 = \left(\frac{4\pi^2}{3}\,\frac{\bar\lambda_2}{\bar\lambda_4^2}\right)^{1/2}$$

Autonomous system analysis

define dimensionless variables

$$x = \frac{\kappa \dot{\phi}}{\sqrt{6}H} \; ; \qquad y = \frac{\kappa\sqrt{V}}{\sqrt{3}H} \; ; \qquad z = \frac{V'}{\kappa V} \qquad \left(\kappa = \sqrt{8\pi G}\right)$$

[Copeland Liddle & Wands '98]

$$\begin{aligned}
\frac{dx}{dN} &= 3x(1-x^2) + \sqrt{\frac{3}{2}}y^2 z + \frac{1}{2}x\eta_{\mathrm{RG}}\frac{d\ln k}{dN} \\
\frac{dy}{dN} &= -\sqrt{\frac{3}{2}}xyz - 3x^2 y + \frac{1}{2}y(\eta_{\mathrm{RG}} + \nu_{\mathrm{RG}})\frac{d\ln k}{dN} \\
\frac{dz}{dN} &= -\sqrt{6}\,x(\eta(z) - z^2) + z\left(-\frac{1}{2}\eta_{\mathrm{RG}} - \nu_{\mathrm{RG}} + \sigma_{\mathrm{RG}}\right)\frac{d\ln k}{dN}
\end{aligned}$$

[Hindmarsh Litim & Rahmede '11]

being $\qquad \sigma_{\mathrm{RG}} = \frac{\partial \ln V'}{\partial \ln k} \qquad ; \qquad \eta(z) = \frac{V''}{\kappa^2 V}$

new terms can be rewritten as

$$\frac{d \ln k}{dN} = \frac{1}{\alpha_{\mathrm{RG}}} \left[\frac{\sigma_{\mathrm{RG}}}{\nu_{\mathrm{RG}}} \sqrt{\frac{3}{2}} xz + 3x^2 \right]$$

where

$$\alpha_{\mathrm{RG}} = \frac{1}{2} \left[\eta_{\mathrm{RG}} + \nu_{\mathrm{RG}} - \frac{\partial}{\partial \ln k} \ln \left(-\frac{\eta_{\mathrm{RG}}}{\nu_{RG}} \right) \right]$$

and it follows that

$$x = \pm \sqrt{1 + \frac{\eta_{\mathrm{RG}}}{\nu_{\mathrm{RG}}}} \quad ; \qquad y = \sqrt{-\frac{\eta_{\mathrm{RG}}}{\nu_{\mathrm{RG}}}}$$

IS RG FP REGIME A COSMOLOGICAL FIXED POINT?

remember that $\nu_{\mathrm{RG}}(\tilde{\phi}) = \mathrm{const}$

- monomial potential $V(\phi) = \lambda_n \phi^n$

$$x = \pm\left(\frac{2-n}{4-n}\right)^{\frac{1}{2}} \; ; \qquad y = \left(\frac{2}{4-n}\right)^{\frac{1}{2}} \; ; \qquad z = -\sqrt{\frac{3}{2}}\, n\, x$$

- trinomial potential $V(\phi) = \lambda_0 + \lambda_2 \phi^2 + \lambda_4 \phi^4$

$$x = \pm\sqrt{\frac{\tilde{\lambda}_0 - \tilde{\lambda}_4\tilde{\phi}^4}{2\tilde{\lambda}_0 + \tilde{\lambda}_2\tilde{\phi}^2}}; \quad y = \sqrt{\frac{\tilde{\lambda}_0 + \tilde{\lambda}_2\tilde{\phi}^2 + \tilde{\lambda}_4\tilde{\phi}^4}{2\tilde{\lambda}_0 + \tilde{\lambda}_2\tilde{\phi}^2}}$$

$$z = \frac{\tilde{\phi}}{\sqrt{2\pi\tilde{G}}}\frac{\tilde{\lambda}_2 + 2\tilde{\lambda}_4\tilde{\phi}^2}{\tilde{\lambda}_0 + \tilde{\lambda}_2\tilde{\phi}^2 + \tilde{\lambda}_4\tilde{\phi}^4}$$

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AND WHAT ABOUT QUASI-CLASSICAL REGIME?

direct evaluation shows that for $\phi \gg \phi_{\rm cl}$ and $A \ll \phi_{\rm cl}$

$$x_{\rm late} \simeq -\sqrt{\frac{8}{3}}\frac{1}{\kappa\phi}, \qquad y_{\rm late} \simeq 1, \qquad z_{\rm late} \simeq \frac{4}{\kappa\phi}$$

and notice that $H \propto \phi^2 \qquad \Rightarrow \qquad k \not\propto H$

Cosmological fluctuations

comoving gauge $T^{0i} = 0 \quad \Rightarrow \quad \delta\phi = \delta k = 0$

spatial part of the metric can be written $g_{ij} = a^2(\tau) e^{-2\mathcal{R}} \delta_{ij}$

second variation of the action gives

$$\Gamma^{(2)}_{\mathcal{R}} = \frac{1}{2} \int d^4x \left((v')^2 - (\partial_i v)^2 + \frac{\theta''}{\theta} v^2 \right)$$

where $\qquad \theta = a\dot{\phi}/H \qquad v = \theta \mathcal{R}$

in fixed point regime $\theta \propto 1/\tau$

$$v''_{\mathbf{p}} + \left(p^2 - \frac{2}{\tau^2} \right) v_{\mathbf{p}} = 0$$

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same form as in standard de Sitter background, solution is

$$v_{\mathbf{p}} = \frac{p\tau - i}{p\tau} e^{-ip\tau}$$

hence, as $\tau \to 0$ from below (late times)

$$|\mathcal{R}_{\mathbf{p}}|^2 \to \frac{1}{(\theta p\tau)^2}$$

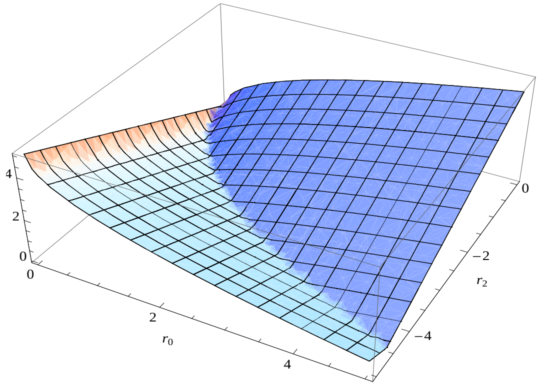
and power spectrum of curvature perturbations can be written

$$\mathcal{P}_{\mathcal{R}}(p) = \frac{1}{24\pi^2} \frac{(1-3x^2)^2}{x^2} \frac{H^2}{m_{\mathrm{Pl}}^2}$$

power spectrum (assuming $\alpha \gg 1$) in terms of couplings

$$\mathcal{P}_{\mathcal{R}} \simeq \frac{32}{3}\pi \tilde{G}^3 \tilde{\phi}^2 (2\tilde{\lambda}_0 + \tilde{\lambda}_2 \tilde{\phi}^2) \quad \Rightarrow \quad n_s = 1$$

$\mathcal{P}_{\mathcal{R}} \ll 1$ can only be achieved if $r_2 \simeq -2\sqrt{r_0}$



smallness parameter $r_2 = -2\sqrt{r_0} + \delta/\tilde{G} \Rightarrow \qquad \mathcal{P}_{\mathcal{R}} = \frac{4}{3}\, \tilde{G}\tilde{\Lambda}\, \delta$

tensor power spectrum

$$\mathcal{P}_h \simeq \frac{32}{3}\tilde{G}^2(2\tilde{\lambda}_0 + \tilde{\lambda}_2\tilde{\phi}^2)\,(-p\tau)^{n_T}$$

where $n_T = -\frac{2}{\alpha-1}$

smallness parameter δ

$$\mathcal{P}_h \simeq \frac{16}{3}\sqrt{\frac{\tilde{G}\tilde{\Lambda}\tilde{\lambda}_4}{2\pi}}\,\delta\,(-p\tau)^{n_T}$$

so that tensor-to-scalar ratio

$$r = \frac{\mathcal{P}_h(p)}{\mathcal{P}_{\mathcal{R}}(p)} \simeq \sqrt{\frac{8\tilde{\lambda}_4}{\pi\tilde{G}\tilde{\Lambda}}}\,(-p\tau)^{n_T}$$

Conclusions

- Exact Renormalisation Group technique indicates that gravity may be asymptotically safe, even with the inclusion of matter fields, like the scalar field considered here
- Fixed point regime of the RG trajectory triggers a phase of power law inflation in the early universe dynamics, and then smoothly approaches classical dynamics at later times
- The vicinity of the trajectory to RG fixed points causes the appearance of cosmological fixed points in the autonomous phase space analysis
- perturbations lying inside the RG length scale can be treated in the standard way and give predictions for the primordial power spectra, as functions of the fixed point values of the couplings

Ongoing work

- ⋆ Widening of the truncation, with the inclusion of *all* operators of canonical dimension $d \leq 4$
- ⋆ Numerical study of cosmological evolution, in order to achieve a complete cosmological history
- ⋆ Production of more realistic predictions for observable quantities, like power spectra