Fractal space-times under the microscope: a RG view on Monte Carlo data

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M. Reuter and F.S., arXiv:1110.5224 [hep-th]

S. Rechenberger and F.S., work in progress

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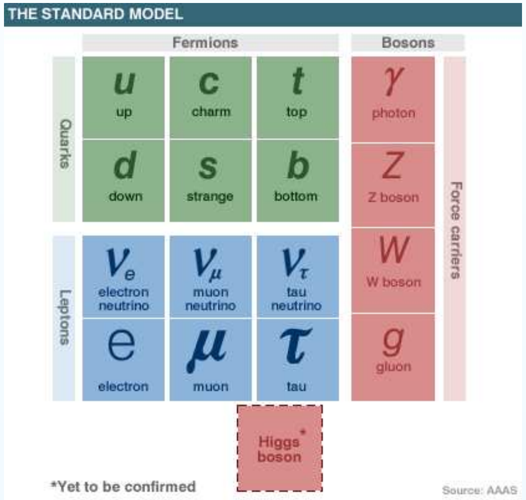
Outline

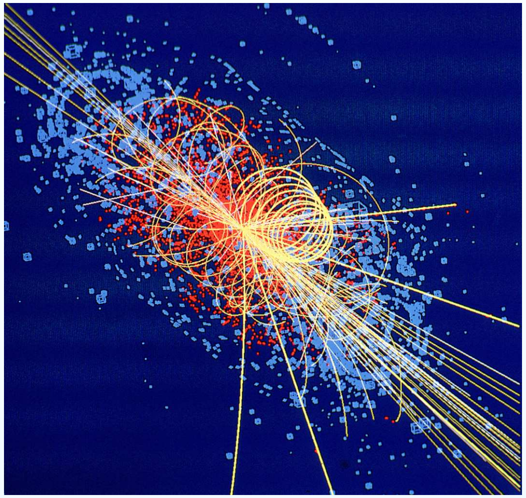
- introduction
- quantifying properties of fractals
- fractal dimensions in QEG
- dimensional flow in the Einstein-Hilbert truncation
- comparison to Monte-Carlo data
- conclusions and outlook

Introduction

standard model of particle physics:

- describes: electromagnetic/strong/weak force + interactions with matter
- theoretical basis: quantum field theory in four-dimensional Minkowski space





works extremely well!

Space-time is four-dimensional, n'est-ce pas?

[B. Müller, A. Schäfer, Phys. Rev. Lett. 56 (1986) 1215]

[M. Haugan, C. Lämmerzahl, Lect. Notes Phys. 562 (2001) 195]

[D. Mattingly, Liv. Rev. Rel. 8 (2005) 5]

- Bounds on dimension from dimensional regularization

$$d_H = 4 - \epsilon$$

  - ϵ: probes fractional space-times which "misses points"

Experimental bounds on $|\epsilon|$:

- anomalous magnetic moment of muon $g-2$:

$$|\epsilon| < 10^{-8}\,, \qquad \ell \approx 10^{-15} m$$

- Lamb shift in hydrogen:

$$|\epsilon| < 10^{-11}\,, \qquad \ell \approx 10^{-11} m$$

- precession of planetry orbits:

$$|\epsilon| < 10^{-9}\,, \qquad \ell \approx 10^{11} m$$

Spontaneous dimensional reduction of space-time?

- Causal Dynamical Triangulations:
  - classical space-time at large distance: $d = 4$
  - diffusion on short scales: effectively two-dimensional
- Renormalization Group Analysis:
  - spectral dimension at NGFP: $d_s = 2$
  - anomalous dimension of $G_{\rm N}$ $\Rightarrow$ two-dimensional graviton propagator
- area-spectrum in Loop-Quantum Gravity:

$$A_j \propto \sqrt{\ell_j^2(\ell_j^2 + \ell_P^2)} \propto \begin{cases} \ell_j^2 & \text{for large area} \\ \ell_j \ell_P & \text{for small area} \end{cases}$$

- string theory at high temperatures
- anisotropic scaling models (Horava-Lifshitz Gravity)
- Strong coupling limit Wheeler-de Witt equation
- $\ldots$

quantifying properties of fractals

Hausdorff or topological dimension

Determined by number N of balls necessary to cover a point-set:

$$N(R) \propto R^{-D}$$

Hausdorff-dimension d_H:

$$d_H = -\lim_{R \to 0} \frac{\log N(R)}{\log R}$$

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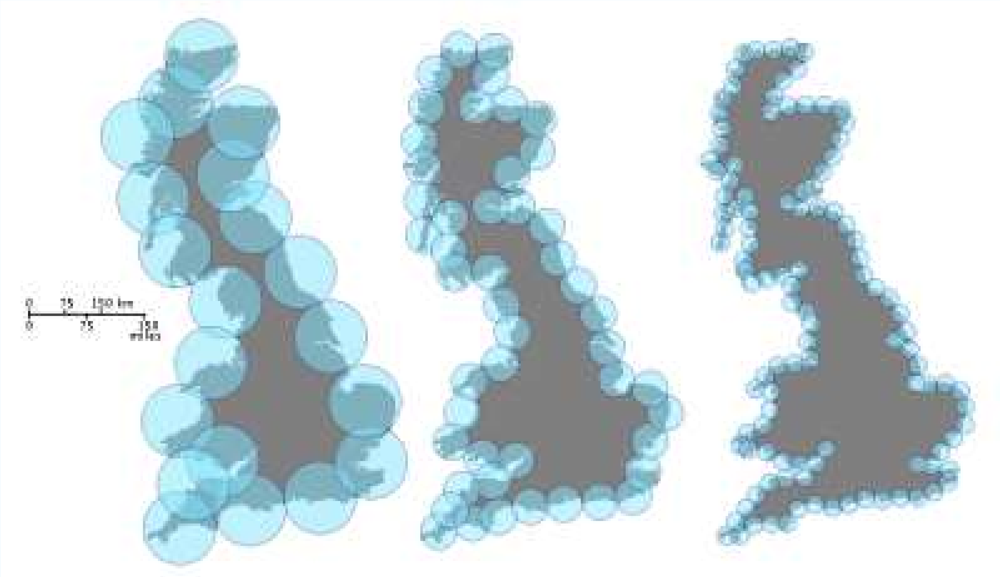
Examples:

- real line: $N(R) \propto R^{-1} \quad \longrightarrow \quad d_H = 1$
- coast-line of England: $d_H \approx 1.2$



Spectral dimension d_s

- Heat-equation: diffusion of scalar test particle on manifold with metric g

$$\partial_T K_g(x, x'; T) = -\Delta_g K_g(x, x'; T)$$

- define averaged return probability

$$\begin{aligned} P_g(T) &\equiv \frac{1}{V} \int d^d x \sqrt{g(x)}\, K_g(x, x; T) \\ &= \frac{1}{V} \mathrm{Tr}\left[\exp(-T\Delta_g)\right] \\ &= \left(\frac{1}{4\pi T}\right)^{d/2} \sum_{n=0}^{\infty} A_n T^n \end{aligned}$$

- generalization: space-time dimension seen by diffusion process

$$d_s = -2 \frac{d \ln P_g(T)}{d \ln T}\Big|_{T=0}$$

Extension to finite random walks: $\mathcal{D}_s(T)$

Walk dimension d_w

characterizes the fractal properties of the trail left by random walk

- probability density for random walk in flat space

$$K(x, x'; T) = (4\pi T)^{-d/2} \exp\left(-\frac{|x - x'|^2}{4T}\right)$$

- average square displacement characteristic for regular diffusion

$$\langle x^2 \rangle = \int d^d x \, x^2 \, K(x, 0; T) \propto T$$

- On fractals: diffusion can be anomalous:

$$\langle x^2 \rangle \propto T^{2/d_w} \Big|_{T=0}$$

definition of walk dimension

Extension to finite random walks: $\mathcal{D}_w(T)$

Alexander-Orbach relation

[S. Alexander, R. Orbach, J. Phys. Lett. (Paris) 43 (1982) L625]

on homogeneous fractals:

$$\frac{d_s}{2} = \frac{d_H}{d_w} \,.$$

- relation between spectral, walk and Hausdorff dimension computing fractal dimensions in QEG

Classical vs. quantum space-times

[O. Lauscher, M. Reuter, JHEP 0510 (2005) 050]

classical space-times from general relativity

$$S^{\rm EH} = \frac{1}{16\pi G_N} \int d^dx \sqrt{g} \, (-R + 2\Lambda)$$

- Einstein equations

$$R_{\mu\nu} = \frac{2}{2-d} \, \Lambda \, g_{\mu\nu}$$

  solution: metric $g_{\mu\nu}$ valid at all length scales

effective quantum space-time: replace $S^{\rm EH} \to$ effective average action $\Gamma_k[g]$

- one-parameter family of equations of motion

$$\frac{\delta \Gamma_k[\langle g_{\mu\nu} \rangle_k]}{\delta g_{\mu\nu}} = 0$$

- solution: metric $\langle g_{\mu\nu} \rangle_k$ seen by physical process with momentum k^2
- proper distance calculated from $\langle g_{\mu\nu} \rangle_k$ depends on k^2

Diffusion processes on QEG space-times

[O. Lauscher, M. Reuter, JHEP 0510 (2005) 050]

basic idea: replace classical return probability by expectation value:

$$P(T) \equiv \langle P_\gamma(T) \rangle \equiv \int \mathcal{D}\gamma \mathcal{D}C \mathcal{D}\bar{C}\, P_\gamma(T)\, e^{-S_{\text{bare}}[\gamma, C, \bar{C}]}$$

$$K(x, x'; T) \equiv \langle K_\gamma(x, x'; T) \rangle$$

- determine expectation values using $\langle \mathcal{O}(\gamma_{\mu\nu}) \rangle \approx \mathcal{O}(\langle g_{\mu\nu} \rangle_k)$:
  - solve EOM of $\Gamma_k[g]$ in Einstein-Hilbert truncation

$$R_{\mu\nu}(\langle g \rangle_k) = \frac{2}{2-d}\, \Lambda_k \, \langle g_{\mu\nu} \rangle_k$$

- scaling relation between metrics at different scales k:

$$\langle g_{\mu\nu}(x) \rangle_k = [\Lambda_{k_0}/\Lambda_k] \langle g_{\mu\nu}(x) \rangle_{k_0}$$

- spectrum of Laplace-operators at different scales

$$\Delta(k) = \left[\Lambda_k / \Lambda_{k_0}\right] \Delta(k_0)$$

Compute the spectral dimension $\mathcal{D}_s(T)$

[O. Lauscher, M. Reuter, JHEP 0510 (2005) 050]

1. spectrum of Laplace-operators at different scales

$$\Delta(k) = \left[\Lambda_k/\Lambda_{k_0}\right] \Delta(k_0)$$

2. solve the k-dependent heat equation

$$\partial_T K(x,x';T) = -\Delta(k) K(x,x';T)$$

- assume Λ_{k_0} small $\Longrightarrow$ flat-space approximation

$$K(x,x';T) = \int \frac{d^dp}{(2\pi)^d}\, e^{ip\cdot(x-x')}\, e^{-p^2 F(p^2) T}\,, \qquad F(p^2) = \Lambda(p)/\Lambda(k_0)$$

3. quantum return probability

$$P(T) = \int \frac{d^dp}{(2\pi)^d}\, e^{-p^2 F(p^2) T}$$

4. spectral dimension for scaling cosmological constant: $\Lambda_k \propto k^\delta$:

$$\boxed{\mathcal{D}_s(T) = \tfrac{2d}{2+\delta}}$$

Compute the walk dimension $\mathcal{D}_w(T)$

1. flat-space approximation of probability density
   - scaling regime $F(p) = (Lp)^\delta$

$$K(x, x'; T) = \int \frac{d^d p}{(2\pi)^d}\, e^{ip\cdot(x-x')}\, e^{-p^{(2+\delta)}\, L^\delta\, T}$$

2. Rescaling $q_\mu = p_\mu T^{1/(2+\delta)}$, $\xi_\mu = (x_\mu - x'_\mu)/T^{1/(2+\delta)}$:

$$K(x, x'; T) = \frac{1}{T^{d/(2+\delta)}} \int \frac{d^d q}{(2\pi)^d}\, e^{iq\cdot\xi}\, e^{-L^\delta q^{2+\delta}}$$

3. $\langle x^2 \rangle$ scales as $T^{2/(2+\delta)}$
4. walk dimension for scaling cosmological constant: $\Lambda_k \propto k^\delta$:

$$\boxed{\mathcal{D}_w(T) = 2 + \delta}$$

Hausdorff dimension of effective QEG space-times

- Volume of d-ball $\mathcal{B}^d$ computed from $\langle g_{\mu\nu} \rangle_k$

$$V(\mathcal{B}^d) = \int_{\mathcal{B}^d} d^d x \sqrt{g_k} \propto (r_k)^d$$

- compare to definition of d_H:

$$d_H = d$$

Hausdorff dimension of effective QEG space-times

- Volume of d-ball $\mathcal{B}^d$ computed from $\langle g_{\mu\nu} \rangle_k$

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- compare to definition of d_H:

$$d_H = d$$

Conclusion:

- QEG space-times are not sponge-like
- fractal properties: dynamical
- QEG space-times satisfy the Alexander-Orbach relation

$$\frac{\mathcal{D}_s}{2} = \frac{d_H}{\mathcal{D}_w} \, .$$

dimensional flow

in the Einstein-Hilbert truncation

Spectral and walk dimension on theory space

Scaling law for cosmological constant

$$\Lambda_k \propto k^{\delta}$$

generalization: $\delta(k)$ as scale-dependent quantity

$$\begin{aligned} \delta(k) &\equiv k\partial_k \ln(\Lambda_k) \\ &= 2 + \lambda_k \beta_\lambda(g, \lambda) \end{aligned}$$

Substitute into fractal dimensions

$$\mathcal{D}_s(g, \lambda) = \frac{2d}{4 + \lambda^{-1}\beta_\lambda(g, \lambda)}$$

$$\mathcal{D}_w(g, \lambda) = 4 + \lambda^{-1}\beta_\lambda(g, \lambda)$$

$\mathcal{D}_s$ and $\mathcal{D}_w$ are autonomous functions of theory space!

β-functions of the Einstein-Hilbert truncation

Einstein-Hilbert truncation: two running couplings: $G(k), \Lambda(k)$

$$\Gamma_k = \frac{1}{16\pi G(k)} \int d^4x \sqrt{g}\,[-R + 2\Lambda(k)] + S^{\rm gf} + S^{\rm gh}$$

- project flow onto G-$\Lambda-$plane

explicit β-functions for dimensionless couplings $g_k := k^2 G(k)\,,\ \lambda_k := \Lambda(k) k^{-2}$

- Particular choice of $\mathcal{R}_k$ (optimized cutoff)

$$k\partial_k g_k = (\eta_N + 2) g_k\,,$$

$$k\partial_k \lambda_k = -(2-\eta_N)\,\lambda_k - \tfrac{g_k}{2\pi}\left[5\,\tfrac{1}{1-2\,\lambda_k} - 4 - \tfrac{5}{6}\,\tfrac{1}{1-2\lambda_k}\eta_N\right]$$

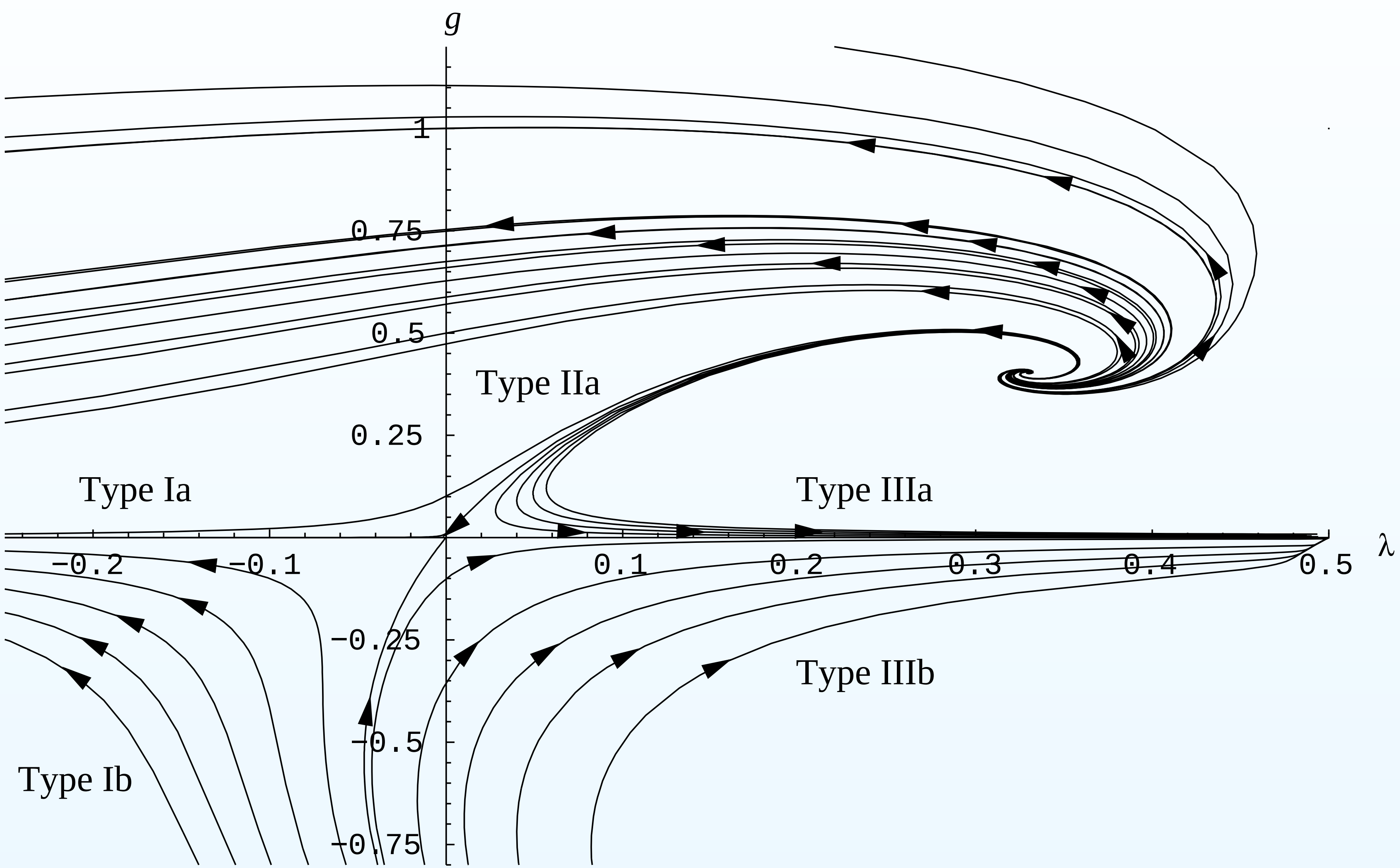
- anomalous dimension of Newton's constant:

$$\eta_N = \frac{gB_1}{1-gB_2}$$

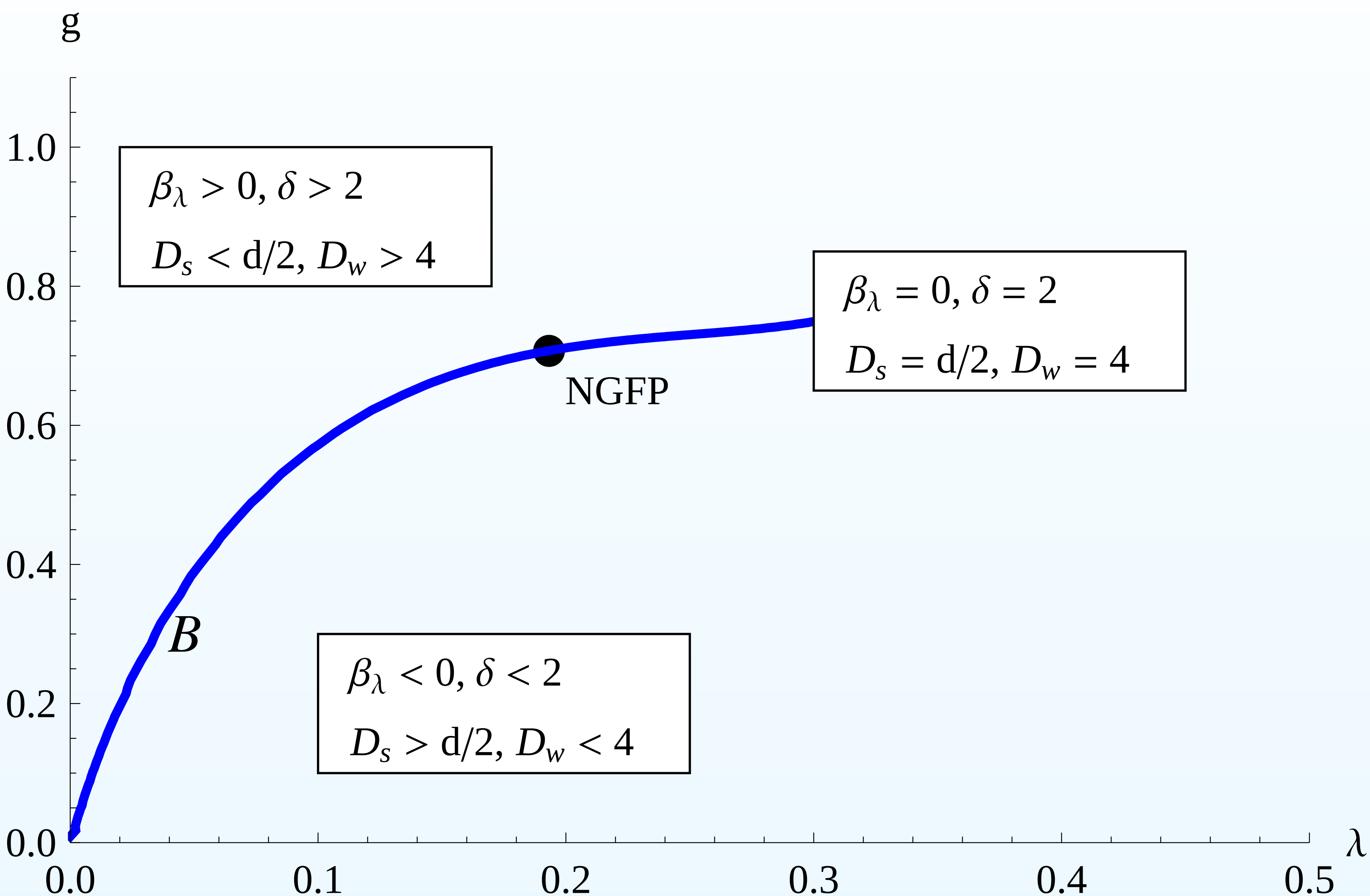
$$B_1 = \tfrac{1}{3\pi}\left[5\,\tfrac{1}{1-2\lambda} - 9\,\tfrac{1}{(1-2\lambda)^2} - 7\right]\,,\ B_2 = -\tfrac{1}{12\pi}\left[5\,\tfrac{1}{1-2\lambda} + 6\,\tfrac{1}{(1-2\lambda)^2}\right]$$

Einstein-Hilbert-truncation: the phase diagram

[M. Reuter, FS, ’01]

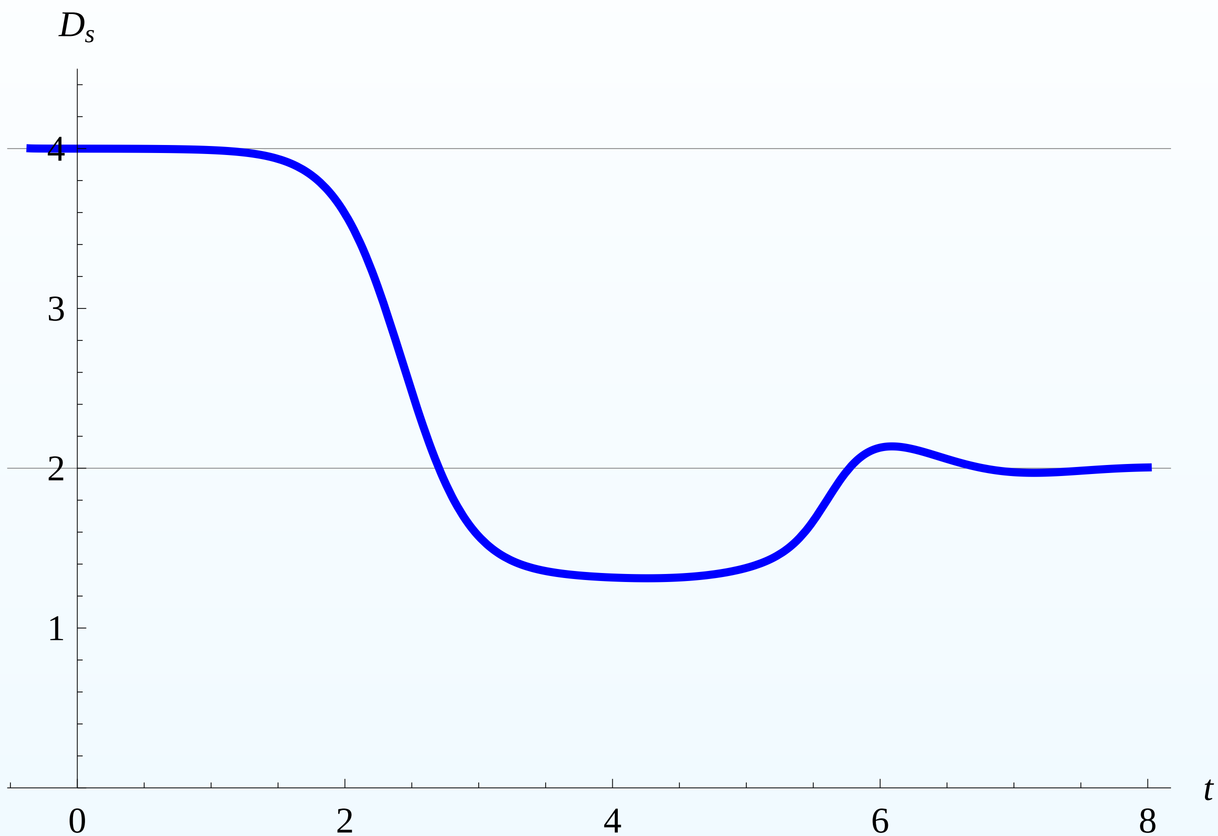


Fractal dimensions on theory space



Spectral dimension $\mathcal{D}_s$ of QEG space-times

Flow of spectral dimension along a typical RG-trajectory



- classical regime: $\mathcal{D}_s(T) = 4$
- semi-classical regime: $\mathcal{D}_s(T) = 4/3$
- NGFP regime: $\mathcal{D}_s(T) = 2$

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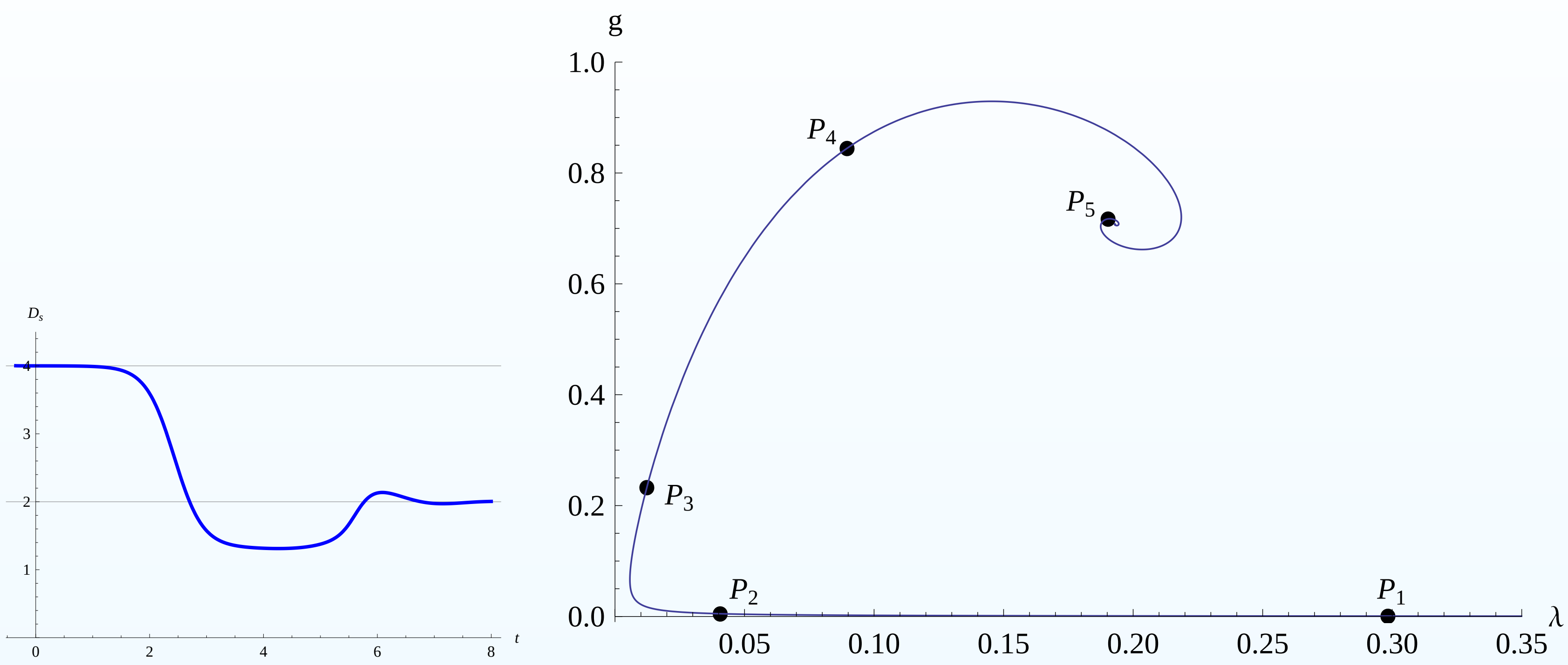

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Spectral Dimension

discrete vs. continuum results

The spectral dimension puzzle

effective QEG space-times [O. Lauscher, M. Reuter '05]

- classical regime $(F(p^2) = 1)$: $\mathcal{D}_s(T) = d$
- NGFP regime $(F(p^2) \propto p^2)$: $\mathcal{D}_s(T) = d/2$

Causal Dynamical Triangulations $(d = 4)$ [J. Ambjorn, J. Jurkiewicz, R. Loll '05]

- classical regime: $\mathcal{D}_s(T) = 4$
- short random walks: $\mathcal{D}_s(T) = 2$

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Causal Dynamical Triangulations $(d = 3)$ [D. Benedetti, J. Henson '09]

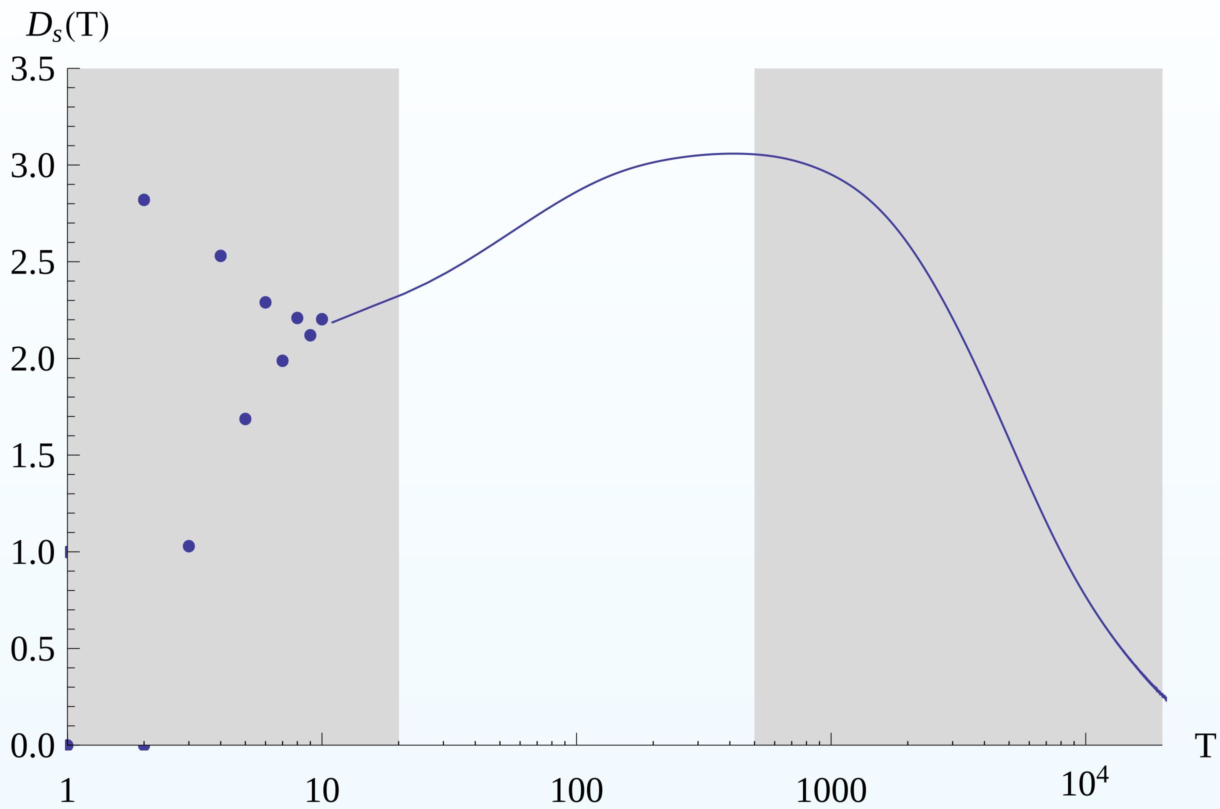
- classical regime: $\mathcal{D}_s(T) = 3$
- short random walks: $\mathcal{D}_s(T) = 2$

Euclidean Dynamical Triangulations $(d = 4)$ [J. Laiho, D. Coumbe '11]

- classical regime: $\mathcal{D}_s(T) = 4$
- short random walks: $\mathcal{D}_s(T) = 1.5$

Spectral Dimension measured in 3-dimensional CDT

[D. Benedetti, J. Henson, Phys. Rev. D 80 (2009) 124036]



	T	≤ 20	oscillations (discrete simplex structure)
$20 \leq$	T	≤ 500	good data
$500 \leq$	T		exponential fall-off (triangulation is compact)

The RG-trajectory underlying the CDT-data

Matching the spectral dimensions of QEG and CDT:

1. integrate β-functions: $g_0, \lambda_0 \mapsto g_k, \lambda_k$
2. substitute RG-trajectory into $\mathcal{D}_s^{\mathrm{QEG}}(T)$:

$$\mathcal{D}_s^{\mathrm{QEG}}(T) \mapsto \mathcal{D}_s^{\mathrm{QEG}}(T; g_0, \lambda_0)$$

3. determine $g_0^{\mathrm{fit}}, \lambda_0^{\mathrm{fit}}$ by minimizing

$$(\Delta \mathcal{D}_s)^2 \equiv \sum_{T=20}^{500} \left(\mathcal{D}_s^{\mathrm{QEG}}(T; g_0^{\mathrm{fit}}, \lambda_0^{\mathrm{fit}}) - \mathcal{D}_s^{\mathrm{CDT}}(T) \right)^2$$

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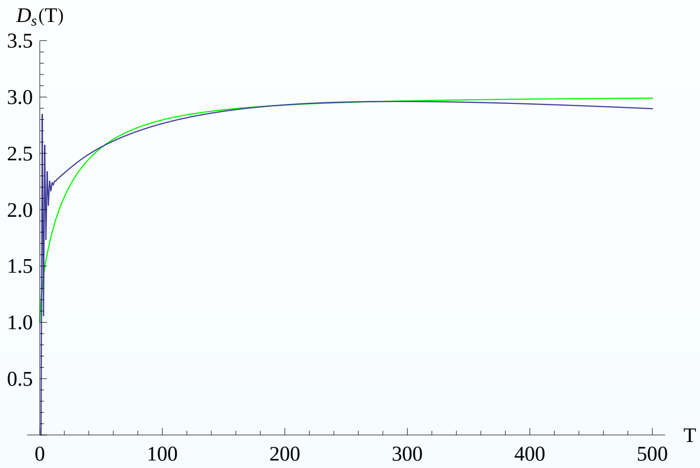
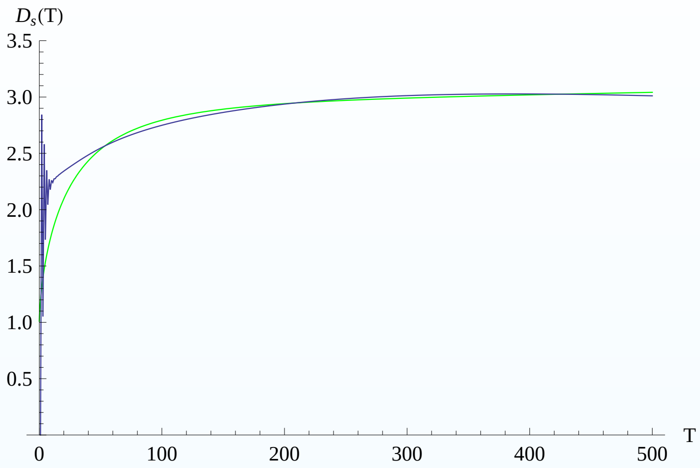
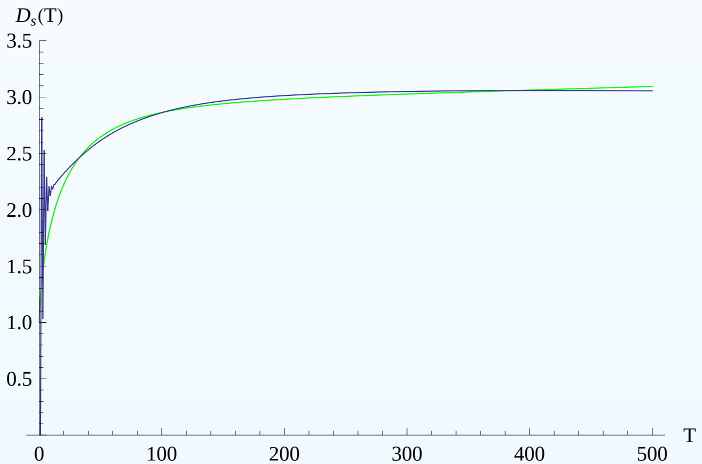
Best-fit values for CDT-data with N simplices:

N	g_0^{fit}	λ_0^{fit}	$(\Delta \mathcal{D}_s)^2$
70k	0.7×10^{-5}	7.5×10^{-5}	0.680
100k	8.8×10^{-5}	39.5×10^{-5}	0.318
200k	13×10^{-5}	61×10^{-5}	0.257

Spectral dimension: comparison

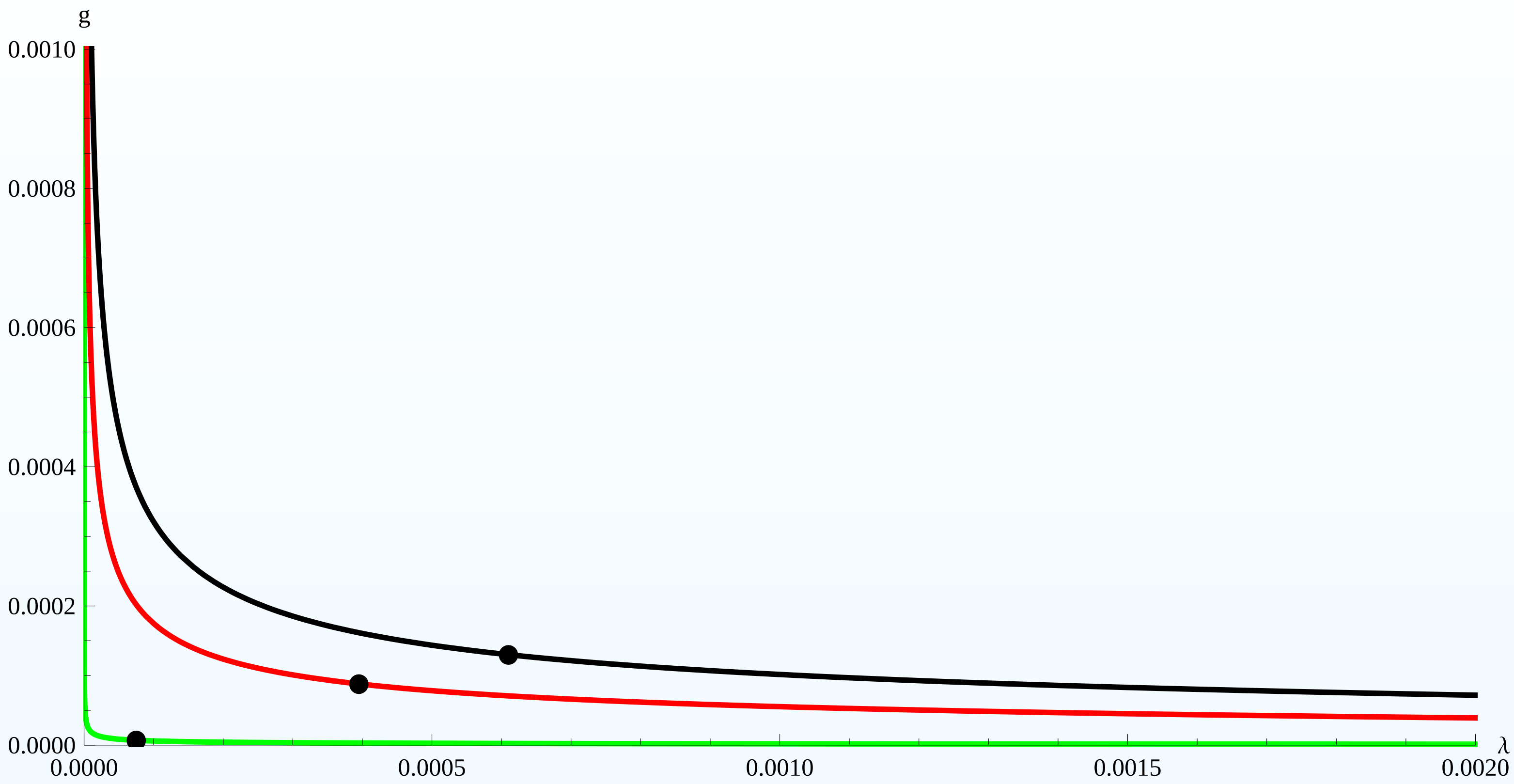






- $\mathcal{D}_s^{\rm QEG}(T; g_0^{\rm fit}, \lambda_0^{\rm fit})$
- $\mathcal{D}_s^{\rm CDT}(T)$ for $N = 70k, 100k, 200k$-simplices

The best-fit trajectories of QEG:

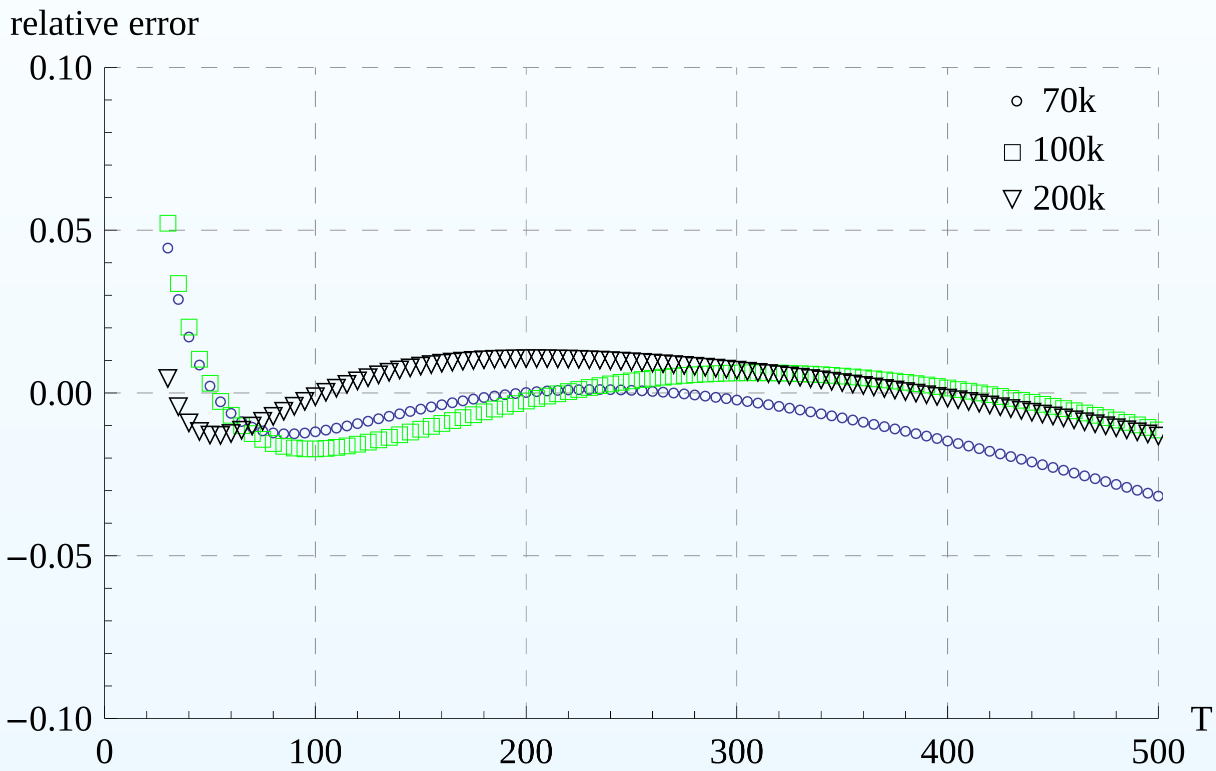


- Distance to GFP increases with increasing N

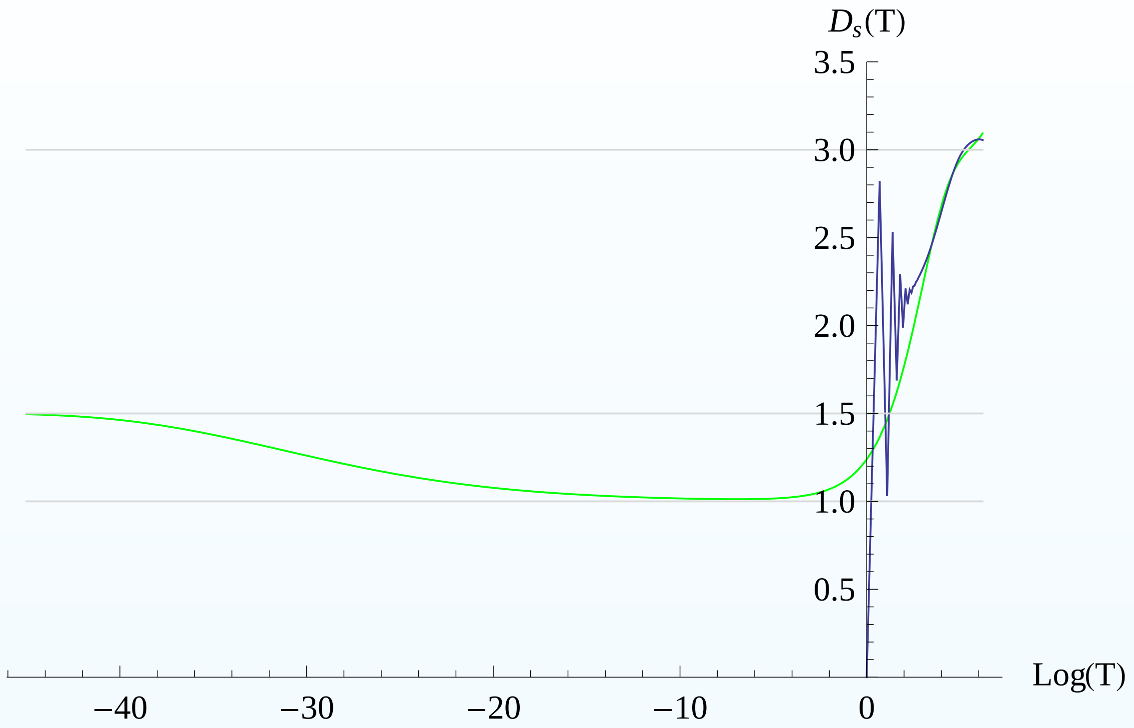
Spectral dimension: fit-quality

relative error captured by residuals:

$$\epsilon \equiv -\frac{\mathcal{D}_s^{\rm QEG}(T; g_0^{\rm fit}, \lambda_0^{\rm fit}) - \mathcal{D}_s^{\rm CDT}(T)}{\mathcal{D}_s^{\rm QEG}(T; g_0^{\rm fit}, \lambda_0^{\rm fit})}$$



Comparing spectral dimensions in $d = 3$



- spectral dimensions obtained in CDT and QEG agree within 1% accuracy
- CDT probes cross-over to semi-classical regime only

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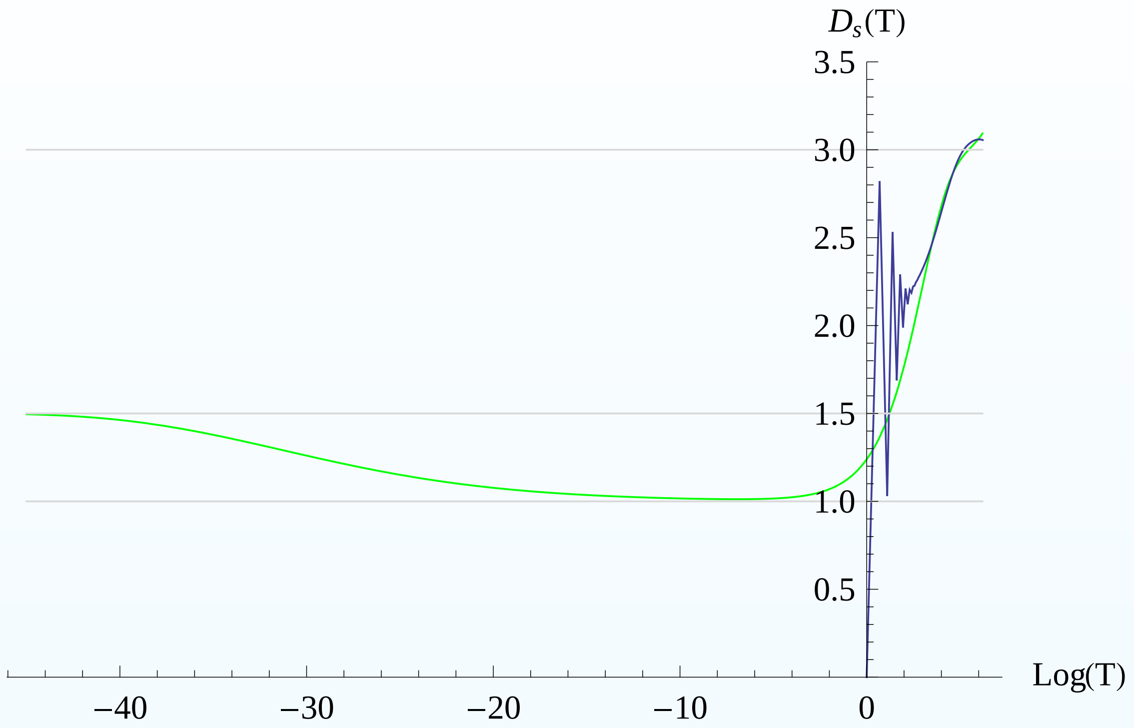

- spectral dimensions obtained in CDT and QEG agree within 1% accuracy
- CDT probes cross-over to semi-classical regime only

resolves puzzle between CDT data and QEG prediction!

Conclusions

Conclusions . . .

spectral, walk and Hausdorff dimensions

- allow to compare quantum structure of space-time in different QG-models

QEG space-times carry multifractal structure

- classical regime: $\mathcal{D}_s(T) = 4$
- semi-classical regime: $\mathcal{D}_s(T) = 4/3$
- fixed-point regime: $\mathcal{D}_s(T) = 2$

connection to Causal Dynamical Triangulations

- CDT-data agrees with QEG prediction to one percent accuracy
- no CDT-data probing semi-classical and fixed-point regime

Conclusions and open questions

QEG space-times carry multifractal structure:

- robustness of plateau-structures (higher-derivative terms)?
- inclusion of matter fields
- spectral dimension on spatial slices

connection to Causal Dynamical Triangulations:

- extending simulations into semi-classical regime

Multi-fractal models of space-time?

Conclusions and open questions

QEG space-times carry multifractal structure:

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Multi-fractal models of space-time?

!!! WORK AHEAD !!!