An effective action for asymptotically safe gravity

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Quantum Einstein-Hilbert theory

- two “running” couplings: $G(k), \Lambda(k)$

$$\Gamma_k^{\mathrm{grav}}[g] = \frac{1}{16\pi G(k)} \int d^4x \sqrt{g}\, \{-R + 2\Lambda(k)\}$$

- project the above “Ansatz" in to a functional-flow-equation (you choose what you prefer!)

- result of several works: non-perturbative β-functions for dimensionless couplings

$$g_k := k^2 G_k\,, \quad \lambda_k := \Lambda_k k^{-2}$$

- Particular choice of $\mathcal{R}_k$ (sharp cutoff)

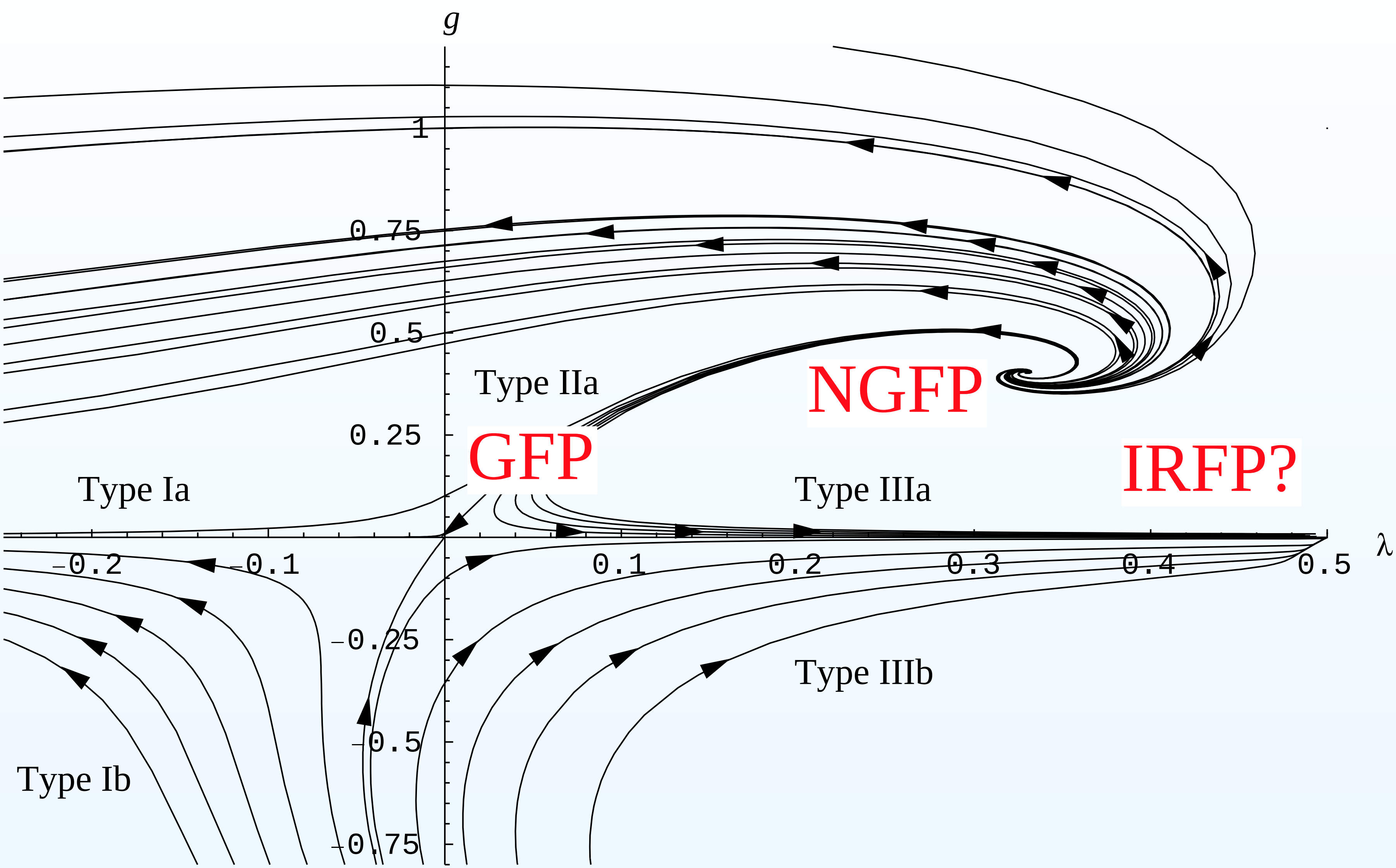
$$k\,\partial_k g_k = (\eta_N + 2) g_k\,,$$

$$k\,\partial_k\,\lambda_k = -\,(2-\eta_N)\,\lambda_k - \tfrac{g_k}{\pi}\left[5\ln(1-2\,\lambda_k) - 2\zeta(3) + \tfrac{5}{2}\eta_N\right]$$

“anomalous" dimension of Newton’s constant:

$$\eta_N = -\frac{2g_k}{6\pi + 5g_k}\left[\tfrac{18}{1-2\lambda_k} + 5\ln(1-2\lambda_k) - \zeta(2) + 6\right]$$

Phase diagram of quantum gravity in the EH-truncation



New Physics

- NGFP and GFP

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\Lambda(k) g_{\mu\nu} + 8\pi G(k)\, T_{\mu\nu}$$

  - $G = G(k)$, $\Lambda = \Lambda(k)$ and $k = k(t)$, or $k = k(H)$
  - with $G(k \to \infty) = 1/k^2$

- IRFP (?): Quantum Gravity at large distances
  - modifications of General Relativity at long distances/late times $\Longleftrightarrow$ origin of dark matter and dark energy
  - AB,M.Reuter PLB, 2002, Bentivegna et al., JCAP 2004, Reuter & Weyer PRD 2004. Rodrigues, arXiv:1203.2286, AB & Carloni, NJP 2012 and arXiv:1112.4613, etc etc...
  - see also: Nagy, Krizsan and Sailer, arXiv:1203.6564

Main difficulty is the “cutoff identification"

- in principle we have

$$k = k(\text{any curvature invariant})$$

  - The cutoff becomes a dynamical variable whose evolution is dictated by RG argument
  - Not all trajectories can be admissible if we specify both $G(k = k(x^\mu)$ and $\Lambda = \Lambda(k = k(x^\mu))$. Ref: AB & MR PLB 2002.
  - Dissipative-fluid description: entropy production in the Early Universe, AB+MR JCAP 2007.

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- possible way out: ask the Bianchi identity do its job and specify only $\Lambda(k = k(x^\mu))$ for instance
  - Reuter & Saueressig JCAP 2005; Koch & Ramirez, CQG 2011, Cai & Easson, arXiv:1202.1285, Hindmarsh, Litim & Rahmede, JCAP 2011 Contillo, Hindmarsh and Rahmede, PRD 2012.

- or use the Brans-Dicke approach: Reuter & Weyer, PRD 2004. AB, Contillo and Percacci, CQG 2011, Cai & Easson JCAP 2011.

Weinberg proposal (PRD 2011) : use an "optimal" cutoff Λ_c

- Asymptotically Safe Inflation
  - More general truncations

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\begin{aligned}
I_\Lambda[g] \quad = \quad & -\int d^4x \sqrt{-\mathrm{Det}g}\Big[\Lambda^4 g_0(\Lambda) + \Lambda^2 g_1(\Lambda) R + g_{2a}(\Lambda) R^2 \\
& + g_{2b}(\Lambda) R^{\mu\nu} R_{\mu\nu} + \Lambda^{-2} g_{3a}(\Lambda) R^3 + \Lambda^{-2} g_{3b}(\Lambda) R R^{\mu\nu} R_{\mu\nu} + \ldots \Big] \; .
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  - Optimal cutoff: *“...radiative corrections just beginning to be important ...and higher terms just beginning to be less important...”*

- Objective: to obtain a de Sitter solution which is unstable but lasts $N > 60$ e-folds

- difficulties: too much fine tuning required, strong dependence on FP quantities
  - Tye & Xu, PRD 2010

Alternative approach: RG improvement at the level of the action

- Work with a cutoff identification at the level of the action.
- pros: general covariance always preserved, f(R) can be mapped onto a scalar field, already uses in QED and QCD
- cons: where do we stop along the IR trajectory?
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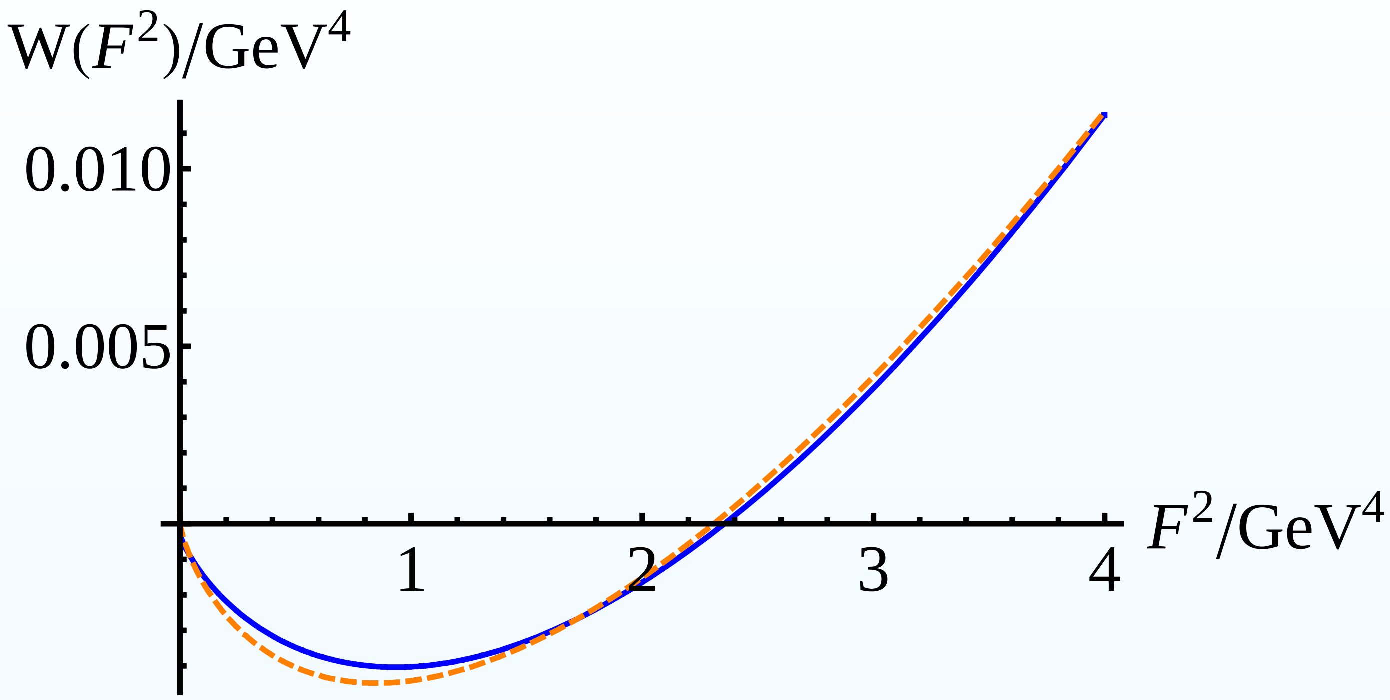
- QCD: leading-log model

$$\mathcal{L}_{\rm eff}^{\rm QCD} = \frac{\mathcal{F}}{2g^2_{\rm run}}, \qquad g^2_{\rm run} = \frac{g^2(\mu^2)}{[1+\frac{1}{4}\, b\, g^2(\mu^2)\log\left(\mathcal{F}/\mu^4\right)]}$$

where $\mathcal{F} = -\frac{1}{2}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc}A_{\mu b}A_{\nu c})^2$, μ is an infrared subtraction scale and b is the usual one-loop renormalization constant, dependent on the number of flavours. (Ref: Matinyan & Savvidy, NPB 1978, Pages & Tomboulis, NPB 1978, Adler NPB 1983...)

RG imp vs ERG



- The effective potential for SU(3) as a function of F^2 (thick blue line), and the one-loop inspired fit to the numerical data of the form $aF^2 \ln bF^2$ (orange dashed line).
  from: Eichhorn, Gies and Pawlowski, PRD 2011

Repeat the same idea with gravity

- Einstein-Hilbert Truncation

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- Consider the linearized flow around the NGFP

$$\begin{aligned}(\lambda, g)^{\mathbf{T}} &= (\lambda_*, g_*)^{\mathbf{T}} + 2\{[\mathrm{Re}C\cos(\theta'' t) + \mathrm{Im}C\sin(\theta'' t)]\mathrm{Re}\, V \\ &+ [\mathrm{Re}C\cos(\theta'' t) - \mathrm{Im}C\sin(\theta'' t)]\mathrm{Im}\, V\}\, e^{-\theta' t}\end{aligned}$$

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- And substitute the linearized flow in to EH Lagrangian with $k^2 \propto R$,

$$\mathcal{L}_{\mathrm{eff}}^{\mathrm{QEG}}(R) = R^2 + bR^2\cos\left[\alpha\log\left(\frac{R}{\mu}\right)\right]\left(\frac{R}{\mu}\right)^{\beta}$$

where: $\alpha = \theta''/2$, $\beta = -\theta' < 0$, $\mu = k_0^2$ and $t = \ln(k/k_0)$.

Note that $\mathcal{L}_{\mathrm{eff}}^{\mathrm{QEG}}(R) \propto R^2$ for $R \to \infty$.

f(R) from RG-improved EH

$$-\frac{1}{\sqrt{|g|}}\frac{\delta S}{\delta g_{\mu\nu}} = \frac{d\mathcal{L}}{dR}\, R_{\mu\nu} - \frac{1}{2}\mathcal{L}\, g_{\mu\nu} - \nabla_\mu \nabla_\nu \frac{d\mathcal{L}}{dR} + g_{\mu\nu}\nabla^\rho \nabla_\rho \frac{d\mathcal{L}}{dR} = 0 \qquad (1)$$

being

$$S = \int d^4x\, \sqrt{|g|}\, \mathcal{L}(R)$$

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being

$$S = \int d^4x\, \sqrt{|g|}\, \mathcal{L}(R)$$

Let us now investigate the physical content of this effective lagrangian in the context of early universe, by considering a spatially flat Friedmann-Robertson-Walker metric in vacuum. In a FRW cosmology with scale factor $a(t)$ we can write both the Einstein tensor $G_{\mu\nu}$ and the Ricci tensor $R_{\mu\nu}$ in terms of the Hubble rate $H(t) = \dot{a}(t)/a(t)$.

FRW cosmology

the (tt)-component and (minus) the trace of (2) become

$$\mathcal{A}(H) = -3(\dot{H} + H^2)\frac{d\mathcal{L}}{dR} + 3H\frac{d\dot{\mathcal{L}}}{dR} + \frac{1}{2}\mathcal{L} \tag{3}$$

$$\mathcal{B}(H) = -6(\dot{H} + 2H^2)\frac{d\mathcal{L}}{dR} + 2\mathcal{L} + 3\frac{d\ddot{\mathcal{L}}}{dR} + 9H\frac{d\dot{\mathcal{L}}}{dR}\,. \tag{4}$$

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For the following analysis, instead of using directly Eq.(6) it is more convenient to eliminate the $\dddot{H}$ term generated by (7) using (6) in order to obtain

$$\dot{H}^2 + 6^\beta b\cos\left[\alpha\ln\left(\frac{6(2H^2+\dot{H})}{\mu}\right)\right]\left(\frac{2H^2+\dot{H}}{\mu}\right)^\beta (2\beta H^4 + (4\alpha^2 - 6$$

$$-\beta(9+4\beta))H^2\dot{H} + (1+\beta)\dot{H}^2 + (\alpha - (1+\beta)(2+\beta))H\ddot{H})$$

$$= 2H(3H\dot{H} + \ddot{H}) + 6^\beta b\alpha\sin\left[\alpha\ln\left(\frac{6(2H^2+\dot{H})}{\mu}\right)\right]\left(\frac{2H^2+\dot{H}}{\mu}\right)^\beta$$

$$(2H^4 - (9+8\beta)H^2\dot{H} + \dot{H}^2 - (3+2\beta)H\ddot{H}) \tag{8}$$

de Sitter solutions

- so that $H = \bar{H} = const$ so that Eq.(8) yields

$$\bar{H} = \sqrt{\frac{\mu}{12}} \exp\left[\frac{1}{2\alpha}\left(\tan^{-1}\frac{\beta}{\alpha} + n\pi\right)\right], \quad n \in \mathbb{Z}$$

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- Weinberg's idea: the relevant question is if these solutions are unstable with characteristic growth time $\gg 1/\bar{H}$ so that inflation comes to an end after a large enough e-folds number $\approx 1/\xi$.
- In order to address this question it is convenient to write

$$H(t) = \bar{H} + \delta H(t)$$

and linearize Eq.(8) around the solutions (9) with $\delta H(t) = \exp(\xi \bar{H} t)$.

Stability of de Sitter solutions

After some manipulations, it is possible to obtain the following stability equation

$$\xi^2 + \xi\, 3\, e^{\frac{n\pi}{2\alpha}} + A = 0$$

where

$$A = -\frac{4\alpha b(-1)^n \left(\alpha^2+\beta^2\right) e^{\frac{\beta \tan^{-1}\left(\frac{\beta}{\alpha}\right)+\pi(\beta+1)n}{\alpha}}}{\alpha b(-1)^n \left(\alpha^2+\beta^2-2\right) e^{\frac{\beta\left(\tan^{-1}\left(\frac{\beta}{\alpha}\right)+\pi n\right)}{\alpha}} - 2\sqrt{\alpha^2+\beta^2}}$$

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- The interesting result of this discussion is that the stability of these inflationary solutions does not depend on the mass scale μ: only the real and imaginary part of the critical exponent and the point in the λ-g-plane, monitored by the constant b, determine the stability of the solution.
- The scheme dependence of the critical exponents turns out to be rather limited, as θ' and θ'' assume values in the ranges $2.1 < \theta' < 3.4$ and $3.1 < \theta'' < 4.3$, respectively for various cutoff functions.

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- But for $n < 0$ A is always negative and one root is unstable. In this case A can be approximated with

$$A \approx -\frac{4\left(\alpha^2+\beta^2\right)e^{\frac{\pi n}{\alpha}}}{\alpha^2+\beta^2-2}$$

- and the unstable root is

$$\xi = \frac{\left(\sqrt{\left(\alpha^2+\beta^2-2\right)\left(25\alpha^2+25\beta^2-18\right)}-3\alpha^2-3\beta^2+6\right)e^{\frac{\pi n}{2\alpha}}}{2\left(\alpha^2+\beta^2-2\right)}$$

Number of e-folds

Therefore:

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- The factor in front of the exponential is always of the order unity and positive for θ' and θ'' in the allowed range as $\alpha = \theta''/2$ and $\beta = -\theta'$
- it is always possible to produce enough e-folds of inflation for n negative enough.

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At last we find

$$1/\xi \approx e^{-n\pi/\theta''}$$

for the number of e-folds. For instance for $n = -3$, $1/\xi \approx 17$ and for $n = -4$ one gets $1/\xi \approx 49$ while for $n = -5$, $1/\xi \approx 140$. It should be stressed that this result is rather remarkable, because it only depends on one “universal" quantity namely the imaginary part of the critical exponent which characterizes the flow around the NGFP.

Conclusions

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- straighforward to include matter field, to extend the model physics

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- Essential role played by the imaginary part of critical exponents
- straighforward to include matter field, to extend the model physics
- main issues :-
  - How to connect with GFP?
  - Universality in the IR and IRFP
  - How to produce CMBR fluctuations?
  - Are the critical exponents real?
  - Understand the role of $k^4 \propto R_{\mu\nu}R^{\mu\nu}$-cutoff.
- go back to the leading-log model again to get inspiration!