$f(R)$ gravity from the renormalisation group

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based on
M. Hindmarsh and I. D. Saltas, arXiv: 1203.3957 [gr–qc]

International Asymptotic Safety Seminar, 23 April 2012

Overview of the talk

- Main idea :

  View the RG improved Einstein-Hilbert action as an effective $f(R)$ model and study its (cosmological) properties.

  - Gain intuition about RG improved cosmology, from a different viewpoint.

- Application in cosmology :

  - UV behavior (*inflation, primordial fluctuations*).
  - Radiation/Matter domination period.
  - IR behavior (late time acceleration).
  - The limit to solar system scales.
  - Stability of the model.

Motivation for modifying GR?

- A big problem of the standard ΛCDM cosmology:
  - *Why is Λ so small?*
  - *Are there any "natural" ways to make it small?*
- Is there any relation between primordial (inflation) and late time acceleration of the Universe?
- Different paths suggested: Modified gravity, scalar fields, stringy scenarios (e.g branes)
  - *Motivation, complicated equations, degeneracies, instabilities, fine tuning problems.*
- A running Λ with cosmic time could be able to provide a dynamical resolution to the cosmological constant problem (*Motivation/Naturalness?*). [1]

[1] I. L. Shapiro, J. Sola, and H. Stefancic, JCAP 0501, 012 (2005),
F. Bauer, Class. Quant. Grav. 22, 3533 (2005),
F. Bauer(2005), arXiv:gr-qc/0512007,
M. Reuter, F. Saueressig, JCAP **0509**, 012 (2005).

RG gravity and scalar–tensor theories

- Scalar tensor (ST) theories of gravity: A scalar field ϕ playing the role of the (running) Planck mass.
  - Fields dynamical and satisfy their own equations of motion.
- RG gravity: $G \to G(k), \Lambda \to \Lambda(k), \ldots, c_i \to c_i(k)$. Couplings promoted to dynamical variables.
  - Running couplings conceptually different from the fields in ST theories. Their running is dictated by an "external" set of equations, the beta functions.
  - Equivalence with Brans–Dicke theories has been studied for the EH truncation, at the level of the equations of motion: [2] Role of Bianchi identities crucial to ensure integrability of the system.
- A cut–off identification at the level of the action: Overcome difficulties with Bianchi identities, covariant formulation of theory, equivalence with ST theories established for higher truncations.[3]

•	Scalar-tensor theories	ϕR	$\Box\phi - \frac{dV}{d\phi} = \kappa^2 T_m$
•	RG improvement of action	$\frac{R-\Lambda(k)}{16\pi G(k)}$	$\nabla^\mu \left(8\pi G(k) T_{\mu\nu}\right) = 0$

[2] M. Reuter & H. Weyer, Phys. Rev. D 69 104022 (2004),
Y. F. Cai, D. A. Easson, arXiv: 1202.1285 [hep-th], (2012).

[3] Andrei V. Frolov, Jun-Qi Guo, arXiv:1101.4995v1 [astro-ph.CO],
A. Bonanno arXiv:1203.1962 [hep-th].

The RG improved Einstein–Hilbert action

- *Effective average action* of gravity at scale k:

$$\Gamma_k[g] = \int \sqrt{-g} d^4x \sum_i^{\infty} c_i(k) R^i$$

Cut-off k defines the typical scale of evaluation: all momenta with $p^2 < k^2$ are suppressed. Effective action satisfies a *Functional Renormalisation Group equation*[4].

- A *family of effective actions* Γ_k smoothly connected from UV ($k \to \infty$) to IR ($k \to 0$), through the system of beta functions for the couplings c_i,

$$k\partial_k c_i(k) = \beta_i(c_1, c_2, \ldots c_j)$$

- The Einstein–Hilbert truncation (in 4D):

$$S = \int d^4x \sqrt{-g} \frac{R - 2\Lambda(k)}{16\pi G(k)}$$

A non-trivial UV fixed point at $(g, \lambda) \simeq (0.016, 0.25)$, and a trivial, "free" one (Gaussian Fixed Point (GFP)) at $(g, \lambda) = (0, 0)$. [5] [6]

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[4] C. Wetterich Phys. Lett. **B 301**, 90 (1993).

[5] M. Reuter, Phys. Rev. D 57 971 (1998),
W. Souma, Prog. Theor. Phys. 102 181 (1999) [hep-th/9907027], [gr-qc/0006008] ,
O. Lauscher and M. Reuter, Phys. Rev. D 65 025013 (2002) [hep-th/0108040] ,
D. F. Litim, Phys. Rev. Lett. 92 201301(2004) [hep-th/0312114],
M. Reuter and F. Saueressig, Phys. Rev. D 65 (2002) 065016 [hep-th/0110054],
R. Percacci and D. Perini, Phys. Rev. D 67 081503 (2003) [hep-th/0207033], Phys. Rev. D 68 (2003) 044018 [hep-th/0304222].

[6] For recent investigations regarding a non-trivial IR fixed point, see
I. Donkin, J. M. Pawlowski (2012) arXiv: [hep-th/1203.4207],
S. Nagy, J. Krizsan, K. Sailer (2012), arXiv:1203.6564v1,
C. Contreras and D. F. Litim, in preparation.

Beta functions and theory space for the 4D Einstein-Hilbert truncation [7]

$$\frac{\partial \lambda}{\partial \ln k} = \beta_\lambda(g,\lambda) \equiv -2\lambda - 2g - \frac{24g\left(3g+\frac{1}{2}(1-3\lambda)\right)}{2g-\frac{1}{2}(1-2\lambda)^2} \qquad \lambda \equiv \frac{\Lambda(R)}{R}$$

$$\frac{\partial g}{\partial \ln k} = \beta_g(g,\lambda) \equiv 2g + \frac{24g^2}{4g-(1-2\lambda)^2} \qquad g \equiv 24\pi G(R) \times R$$

Gaussian Fixed Point regime

$$\Lambda = \frac{1}{2}\lambda_T \frac{k^4}{k_T^2} + \frac{1}{2}\lambda_T k_T^2, \qquad G = \text{const.} = m_p^{-2}$$

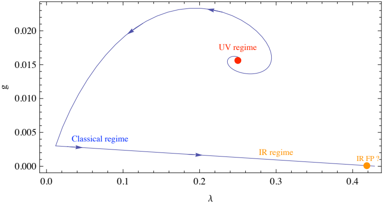
GR regime: Trajectory very close to the GFP, $g \sim \lambda \ll 1$

Smallness of $g \sim \lambda$:

$\Rightarrow$ Large hierarchy: $k_{Turn} = \sqrt{g_{Turn}}\, m_p \ll 1$

$\Rightarrow$ Long classical regime: $k_{term} = \sqrt{g_{Turn}}\, k_{Turn}$

$\Rightarrow$ Smallness of Λ comes for free: $\frac{\Lambda}{m_p^2} = g_{Turn}^2 \ll 1$



A single fine tuning ensures both existence of classical regime and smallness of Λ.

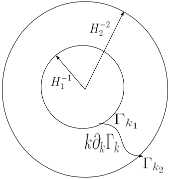
(see for example M. Reuter, H. Weyer, JCAP 0412:001 (2004).

[7] Here, they are calculated in the optimised cut-off of Ref. D. F. Litim, Phys. Lett. B 486 92 (2000).

How to identify the RG cut-off scale k in cosmology?

- Either at the level of the equations of motion or the action, k should be identified as some function of cosmic time and space, x^{μ}.
- In FRW cosmology, the cut-off scale can be identified as some (physically meaningful) function of cosmic time t only.
- One expects that cosmological properties of a given action should not depend on the particular identification.

- $k \rightarrow k(t)$:
- Monotonically decreasing function
- Consistency with Bianchi identities



What are the options?

Average energy of the Universe	$k \sim \quad k_B T(t)$
Horizon of the observable Universe	$k \sim \quad H^{-1}(t)$
Scalar curvature	$k \sim \quad R$
Use Bianchi identities	$\nabla^{\mu}\left(8\pi G\left(k(t)\right) T_{\mu\nu}\right) = 0$

⚠ Not all identifications will satisfy the Bianchi identities.[8]

[8] A. Bonanno and M. Reuter, Phys. Rev. D **65**, 043508 (2002).,
M. Reuter & H. Weyer, Phys. Rev. D 69 104022 (2004).
M. Reuter and F. Saueressig, JCAP **0509**, 012 (2005),
A. Babic, B. Guberina, R. Horvat and H. Stefancic, Phys. Rev. D **71**, 124041 (2005).
M. Hindmarsh, D. Litim & C. Rahmede, JCAP 1107 019 (2011).

$f(R)$ model from the Renormalisation Group

$$\mathcal{L}_k[g] = \frac{R - 2\Lambda(k)}{16\pi G(k)}$$

RG improved Einstein-Hilbert action

$\downarrow$ $k^2 = \rho R$, $g(R) \equiv \rho R \times G(R)$, $\lambda(R) \equiv \frac{\Lambda(R)}{\rho R}$

$$\mathcal{L}_{f(R)}[g] = \rho R^2 \frac{(1 - 2\rho\lambda(R))}{\tilde{\kappa}^2 g(R)}$$ [9]

Effective $f(R)$ model

The RG improved Einstein-Hilbert action can be re-expressed as an $f(R)$ model:

- Quantum corrections of the RG improved action, included in the running of $G(k)$, $\Lambda(k)$, are now "absorbed" in a non-linear, effective $f(R)$ model.
- $f(R)$ model behaves as R^2 gravity on an RG fixed point, where g, λ constant.
- Dimensionless parameter $\rho \equiv k^2/R$ controls how curvature "follows" the cut-off k; in principle an arbitrary parameter, but constrained by cosmological and solar system considerations.

[9] $\tilde{\kappa}^2 \equiv 192\pi^2$.

Fundamentals of $f(R)$ gravity

$f(R)$ gravity is the simplest non-linear extension of Einstein–Hilbert gravity, where R is promoted to a function $f(R)$. [10]

$$f(R) \equiv \frac{R}{16\pi G} + \underbrace{c_2 R^2 + c_3 R^3 \dots}_{\text{primordial acceleration}} + \underbrace{d_1 R^{-1} + d_2 R^{-2} + \dots}_{\text{late time acceleration}}$$

- Graviton and massive scalar $\quad \Box f_R(R) + \frac{dV_{\rm eff}(R)}{df_R} = \frac{\kappa}{3} T_{\rm (m)}$
- Graviton and scalar not a ghost $\quad f_R(R) > 0$
- Scalaron not a tachyon $\quad m^2_{\rm eff} \equiv \frac{d^2 V_{\rm eff}}{df_R^2} \equiv \frac{f_R - R f_{RR}}{3 f_{RR}} > 0$

Remember : $\quad f(R) \equiv f(R, g(R), \lambda(R))$.

- Stability conditions can be expressed in terms of the couplings g, λ, defining characteristic curves on the theory space.
- Stability requirements constraint the allowed range of parameter ρ.

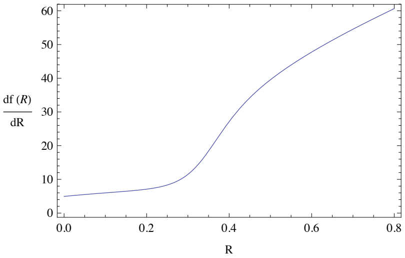
[10] T. P. Sotiriou and V. Faraoni, Rev. Mod. Phys. **82**, 451–497 (2010), A. De Felice and S. Tsujikawa, Living Rev. Rel. **13**, 3 (2010).

The form of the effective $f(R)$ model

- The RG effective $f(R)$ model is defined through the evolution of the couplings g, λ:

$$g(R), \lambda(R) \quad \mapsto \quad f(R) \equiv \rho R^2 \left(\frac{1-2\rho\lambda(R)}{g(R)} \right)$$

- We can understand the general behavior of the function $f(R)$ by evaluating along a trajectory $g(R), \lambda(R)$ for a set of "toy" initial conidtions: $R_{\rm min} = 8 \times 10^{-5}$, $R_T = 5 \times 10^{-3}$, $R_{\rm max} = 50, \lambda(R_T) = 10^{-2}$, $g(R_T) = 10^{-3}, \rho = 1$,



- UV FP regime : $f(R) \sim R^2 \left[1 + cos \left(log \left(\frac{R}{\mu} \right) \right) \left(\frac{R}{\mu} \right)^{\beta} \right]$[a]

- Intermediate regime : $f_R \sim R$ $(f(R) \sim R^2)$.

- GFP and IR regime : $f_R \sim const.$. $f_{RR} > 0$ reflects positivity of scalaron's mass, $m^2_{\rm eff,0} = \frac{f_R - R f_{RR}}{3 f_{RR}} > 0$

[a] A. Bonanno arXiv:1203.1962 [hep-th].

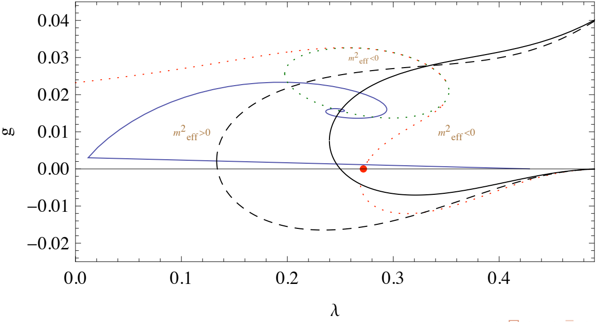
Cosmological viability from UV to IR: Requirements

- Existence of an unstable de Sitter period in the UV.
  - Non-trivial de Sitter, $Rf_R(R) - 2f(R) = 0 = 2\rho^2 g\beta_\lambda + (1 - 2\rho\lambda)\beta_g$.
- Existence of a viable radiation/matter domination period.
  - GR-like evolution, $f_{RR}(R) \simeq 0$, $g, \lambda \ll 1$ (GFP regime).
- Accelerated period for the Universe at late times, (effectively) produced by a vacuum energy $\Lambda/m_{\mathrm{p}}^2 \sim 10^{-120}$.
- Have all above periods dynamically connected in a viable way.
- Evasion of solar system tests.
  - Scalaron very massive $m_{eff}^2(g, \lambda) \gg R$.
- Agreement with the amplitude of primordial fluctuations observed in CMB radiation, as well as with large scale structure.

Does our RG effective $f(R)$ model satisfy above conditions?

Cosmological considerations on the theory space [11]

- *de Sitter line (solid black)*:
  - Passes *through the UV RG fixed point*: *Infinite number* of de Sitter points in the vicinity of the fixed point, due to complex eigenvalues.
  - UV RG fixed point is "hidden" by the outer dS point and cannot be accessed cosmologically.
  - Outer UV dS is unstable, IR one is stable.
- *Slow roll line (dashed black)*:
  - End of inflation, through violation of the slow roll condition $\frac{\dot{H}}{H^2} \ll 1$.
- *Vanishing (dotted green)/Divergence (dotted orange) of m^2_{eff} line*:
  - Scalaron mass vanishes on the UV RG fixed point.
  - Positivity of m^2_{eff} at classical scales.



[11] Here, analysis is done for $\rho = 1$.

Background cosmological dynamics: From UV to IR

We can study cosmological evolution by "RG-improving" the dynamical system for $f(R)$ gravity found before. [12]

$$x_1' = -1 - x_3 - 3x_2 + x_1^2 - x_1 x_3 + x_4$$
$$x_2' = \frac{x_1}{x_3} - x_2(2x_3 - x_1 - 4)$$
$$x_3' = \frac{-x_1 x_3}{m(r)} - 2x_3(x_3 - 2)$$
$$x_4' = -2x_3 x_4 + x_1 x_4$$

$$g' = \frac{\beta_g}{2R}\left(\frac{f^2}{f_R^2 R - f_R f - f_{RR} f R}\right)\left(\frac{x_3' x_2 - x_2' x_3}{x_2^2}\right)$$
$$\lambda' = \frac{\beta_\lambda}{2R}\left(\frac{f^2}{f_R^2 R - f_R f - f_{RR} f R}\right)\left(\frac{x_3' x_2 - x_2' x_3}{x_2^2}\right)$$

$$\Omega_m \equiv \frac{\kappa^2 \rho_m}{3H^2 f_R} = 1 - x_1 - x_2 - x_3 - x_4$$

Dimensionless dynamical variables

$$x_1 = \frac{-\dot{f}_R}{H f_R}, \quad x_2 = \frac{-f}{6H^2 f_R}$$
$$x_3 = \frac{R}{6H^2}, \quad x_4 = \frac{\kappa^2 \rho_r}{3H^2 f_R}$$

Model definition through $m = m(r)$

$$m \equiv \frac{R f_{RR}}{f_R}$$
$$r \equiv -\frac{R f_R}{f} = \frac{x_3}{x_2}$$

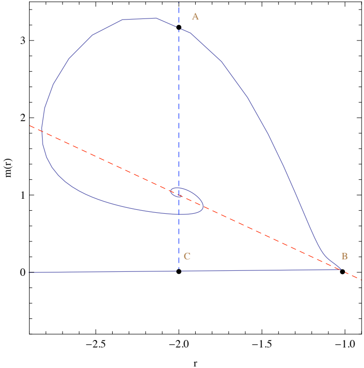
Remember : $f \equiv f(R, g, \lambda)$

- Function $m = m(r)$ determines form of $f(R)$ and needed to close the system!

[12] L. Amendola, R. Gannouji, D. Polarski & S. Tsujikawa, Phys.Rev. D 75 083504 (2007).

- The behavior of the function $m = m(r)$ is able to characterize the asymptotic behavior of the $f(R)$ model: *fixed points and their stability*.

Reminder: $m \equiv \frac{Rf_{RR}}{f_R}$, $r \equiv -\frac{Rf_R}{f}$



de Sitter points

$\bullet P_{A,C} = (0, 1, -2, 0)$,

Stability: $0 < m(r)|_{r=-2} < 1$.

Matter point

$\bullet P_B = \left(\frac{3m_0}{m_0+1}, -\frac{4m_0+1}{2(m_0+1)^2}, \frac{4m_0+1}{2(m_0+1)}, 0 \right)$,

Stability: $m(r)\big|_{r=-2} \simeq +0$,

$$\left.\frac{dm(r)}{dr}\right|_{r=-2} > -1.$$

- In the presence of radiation, i.e $x_4 \neq 0$, a radiation point exists in the vicinity of the matter one - see Ref. L. Amendola, R. Gannouji, D. Polarski & S. Tsujikawa, Phys.Rev. D 75 083504 (2007).
- Numerical investigation shows that viable background evolution requires that

  $0.9 \lesssim \rho \lesssim 1.1$, otherwise Universe does not evolve through matter domination and late time de Sitter.

Solar and astrophysical scales

- It is important to ensure that the model evades solar system tests.
- Solar and astrophysical scales are recovered in the vicinity of the Gaussian Fixed Point regime, where the beta functions take the linear form

$$\frac{\partial \lambda}{\partial \ln k} = -2\lambda + 2\alpha g,$$
$$\frac{\partial g}{\partial \ln k} = 2g.$$

- $f(R)$ model can be expanded as

$$f(R) = f(R_0) + \left. f_R \right|_{R_0} (R - R_0) + \frac{1}{2} \left. f_{RR} \right|_{R_0} (R - R_0)^2$$

...and matched with the renormalisation conditions

$$\left. \frac{R f_R - f}{2 f_R} \right|_{R_0} = \Lambda_0 \qquad \left. f_R \right|_{R_0} = \frac{\tilde{\kappa}^2}{8\pi G_0} \qquad \Bigg| \qquad f(R) \simeq \frac{\tilde{\kappa}^2}{G_0} (R - 2\Lambda_0) + 6(2-\rho)\rho(R - R_0)^2$$

Solar and astrophysical constraints

- The scalaron mass in the GFP linear regime ($g \sim \lambda \ll 1$) is of the *Planck order*

$$m^2_{\mathrm{eff},0}(g,\lambda) \simeq \frac{1}{36(2-\rho)}\frac{R_0}{g} \simeq \frac{1}{36(2-\rho)}\frac{\tilde{\kappa}^2}{8\pi G_0}$$

  Heavy scalaron mass prevents from observable deviations from GR.

- *Positivity of scalaron's mass* at solar scales requires that

$$\rho < 2$$

- Remember that at solar system scales the couplings g, λ acquire a *tiny value*

$$k^{-1}_{\mathrm{sol}} \sim R^{-1/2}_{\mathrm{sol}} \sim 1\mathrm{AU} \Rightarrow g_{\mathrm{sol}} \simeq R_{\mathrm{sol}} \times G_{\mathrm{sol}} \simeq 10^{-92},$$

$$g\lambda \sim G_0\Lambda_0, \lambda_{\mathrm{sol}} \ll 1.$$

Primordial inflation

- A speculated period of accelerating expansion of the Universe at very early times with (quasi-) de-Sitter expansion $a(t) \sim e^{Ht}$, as long as the slow roll condition is satisfied

$$\epsilon \equiv \frac{\dot{H}}{H^2} = -\frac{d \ln H}{dN} \ll 1$$

- In our scenario, inflation is expected to happen in the vicinity of the UV RG fixed point, where $g, \lambda \sim 10^{-1}$ [13].

- Quantum fluctuations of the field Φ during inflation are the seeds for the large scale matter fluctuations today. The power spectra of scalar and tensor fluctuations are constrained through CMB observations [14]

$$\mathcal{P}_s \simeq 2 \times 10^{-9} \;, \quad \mathcal{P}_g \lesssim 0.2 \mathcal{P}_s$$

What does the RG effective $f(R)$ model predict for the primordial fluctuations?

[13] It is important to note that, for an acceptable radiation/matter era to exist, one has to ensure that a viable reheating period exists after inflation has ended.

[14] E. Komatsu et. al, Astrophys. J. Suppl.192 18 (2011).

Einstein frame action

It is useful for inflationary calculations to express the original $f(R)$ action *first* in the *Jordan frame* through the auxiliary scalar σ

$$S = \frac{1}{2\tilde{\kappa}^2} \int \sqrt{-g}\,[f'(\sigma)R - (f'(\sigma)\sigma - f(\sigma))]$$

and then through two field redefinitions to the Einstein frame

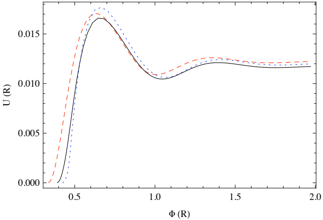
$$\tilde{g}_{\alpha\beta} = \frac{8\pi G_0 \phi}{\tilde{\kappa}^2} g_{\alpha\beta}$$

$$\phi = \phi_0 \exp\left(\sqrt{\frac{16\pi G_0}{3}}\Phi\right)$$

$$\widetilde{S} = \int d^4x \sqrt{-\tilde{g}} \left(\frac{1}{16\pi G_0}\widetilde{R} - \frac{1}{2}(\nabla\Phi)^2 - U(\Phi)\right)$$

$$U(R) = \frac{\tilde{\kappa}^2}{2\,(8\pi G_0)^2} \frac{R f_R(R) - f(R)}{f_R(R)^2}$$

$$\Phi(R) = \sqrt{\frac{3}{16\pi G_0}} \ln f_R(R).$$



- Field Φ starts from top of the potential (unstable UV dS), and evolves to the left towards small field values.

An analytic inflationary model

We can first get an idea about primordial fluctuations by using an analytic approach.

- We can use the $f(R)$ model calculated at the linearised regime ($k \ll m_{\rm p}$) and extrapolate up to $k \sim m_{\rm p}$,

$$f(R) = \frac{R - 2\Lambda_0}{G} + 6\,(2-\rho)\,\rho R^2 .$$

- Inflation could be then realised as R^2 inflation [15].
- In the usual R^2 inflationary scenario, the coupling of the R^2 term, m_p^2/M^2, is constrained to be $m_p^2/M^2 \simeq 10^{11}$ for observationally viable inflation.
- However, for $\rho \sim 1$ (classical regime constraint) , it turns out that $\mathcal{O}(1) \times R^2$ and inflation cannot be viable.

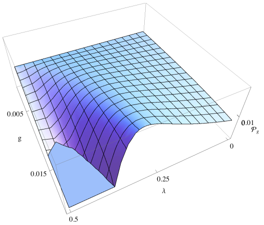
[15] Starobinsky A A 1980 *Phys. Lett.* B **91** 99.

The inflationary power spectra

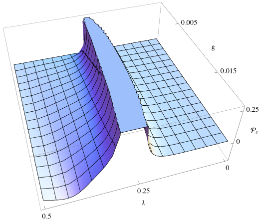
$$\mathcal{P}_s\left(\Phi(g,\lambda)\right) = \frac{128\pi}{3}\left.\frac{U^3}{m_p^2 U_\Phi^2}\right|_{k=aH}, \qquad \mathcal{P}_g(\Phi(g,\lambda)) = \frac{128}{3}\left.\frac{U}{m_p^4}\right|_{k=aH}$$

- Inflation is expected at high energies, i.e in the vicinity of the UV RG fixed point, where $g\lambda = G\Lambda \sim 10^{-2}$, implying that $P_g \sim 10^{-2}$.
- For $\rho \sim 1,\ P_s \sim P_g \sim 10^{-2}$; fluctuations *too large* to be observationally viable.
- Possible solutions?

$\mathcal{P}_s\left(g,\lambda\right)$



$\mathcal{P}_g(g,\lambda)$



Conclusions

- The covariant *cut-off identification* $k^2 = \rho R$ provides us with a very useful tool: An effective $f(R)$ model from an RG improved action.

- *The RG effective $f(R)$ model in the EH truncation* has very interesting properties: Behaves as R^2 gravity around a non-trivial fixed point, exhibits an infinite number of dS points in the UV, a stable dS point in the IR, a viable radiation/matter domination and is able to evade solar system tests.

- Solar system and cosmological requirements *constrain* dimensionless parameter ρ to be $\rho \sim 1$.

- *Primordial inflation* gives very large fluctuations, $G\Lambda \sim \mathcal{O}(1)$, indicating that more ingredients in the action are needed.

- *Open questions*: How generic are the features found? (Study of higher truncations, RG improvement of matter fields, inclusion of scalar field(s) in the action.)

Thank you!