The local potential approximation in quantum gravity

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based on arXiv:1204.3541 (with Francesco Caravelli)

Outline

1. Motivations and general idea

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2. Some technical details and results

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3. Conclusions and outlook

Motivations

- Many beautiful results for asymptotic safety
- In particular, for AS in pure gravity: polynomial truncations

$$\sum_{j=0}^{n} a_j R^j \,, \quad n = 6, 8, 10$$

[Codello, Percacci, Rahmede; Machado, Saueressig; Bonanno, Contillo, Percacci]

$\rightarrow$ Importance for the asymptotic safety conjecture:

  - Test of NGFP beyond EH
  - Number of relevant directions

Some natural questions

1. Do results from truncations converge?

2. How to systematically reject spurious fixed points?

3. Unitarity?

   Any truncated effective action contains higher derivatives

   (heuristic argument [Floreanini,Percacci; DB,Machado,Saueressig] suggests that ghosts might be decoupled in an asymptotically safe theory of gravity, but it is far from conclusive)

4. Non-local IR modifications of gravity?

All these questions suggest to look for approximation schemes which retain an infinite number of terms
$\Rightarrow$ restore the word “functional” in the FRGE

LPA – Scalar field theory

$$\Gamma_k[\phi] = \int d^d x \Big[V_k(\phi) + \partial^\mu \phi \partial_\mu \phi \Big]$$

$\Rightarrow$ Obtain flow of the effective action for constant field

- Compare to truncations: $\sum_{n=0}^{N} u_{2n}(k)\phi^{2n}$ vs. $V_k(\phi)$

	truncations	LPA
flow	ODE	PDE
fixed points	algebraic	ODE

Increased technical effort pays off:

- Derivative expansion typically gives more reliable results than truncations
  (Spanning an infinite-dimensional subspace of the theory space)
- It provides a criterion for discerning true fixed points from spurious ones
  - All but a small discrete set of initial conditions lead to singularities at finite ϕ
  - Fixed points correspond to globally defined solutions

[Hasenfratz&Hasenfratz; Felder; Morris; ...]

An LPA for gravity?

- Expansion in curvature invariants is obviously not an expansion in powers of the field (the metric)
- However it is also not properly a derivative expansion:
  - R^3 Lagrangian propagates as many dof as R^2 Lagrangian
  - it leads to algebraic equations for fixed points, as in field expansions
- The simplest gravitational Lagrangian that can be written without restricting to any specific function is that of an $f(R)$ theory
  - it contains the least number of derivatives among Lagrangians with generic functions (2nd order for TT (spin 2) component; 4th order for scalar component (which if 2nd order is non-propagating))

$\Rightarrow f(R)$ as the LPA of gravity

The special role of maximally symmetric spacetimes

- Decompose the Riemann tensor into its irreducible components:

$$R_{\mu\nu\rho\sigma} = R \oplus S_{\mu\nu} \oplus C_{\mu\nu\rho\sigma}$$

where

$$S_{\mu\nu} = R_{\mu\nu} - \frac{1}{d} g_{\mu\nu} R\,, \quad C_{\mu\nu\rho\sigma} = \text{Weyl tensor}$$

- Express action in terms of irreducible components and their derivatives:

$$\bar{\Gamma}[g] = \int d^d x \sqrt{g}\,\{R + ... + R^n + ... + R\nabla^2 R + C^3 + S^4 + C^2 S^2 + ...\}$$

- Maximally symmetric spacetime: $\nabla_\mu R = S_{\mu\nu} = C_{\mu\nu\rho\sigma} = 0$

$$\bar{\Gamma}[g] = \int d^d x \sqrt{g}\,\{f(R) + \text{things which are zero for MSS}\}$$

- Next order? E.g. Einstein spacetime: $\nabla_\mu R = S_{\mu\nu} = 0$, $C_{\mu\nu\rho\sigma} \neq 0$
  (used for "EH+$R^2 + C^2$" [DB,Machado,Saueressig])
- Einstein/near-MSS expansion:

$$\bar{\Gamma}[g] = \int d^d x \sqrt{g}\,\{f(R) + f_1(R) C^2 + O(C^3)\}$$

Going beyond polynomial truncations

$f(R)$ functional RG

- The ansatz is

$$\bar{\Gamma}_k = Z_k \int d^d x \sqrt{g} f_k(R) = k^d \int d^d x \sqrt{g} \tilde{f}_k(\tilde{R})$$

(All $\tilde{}$ quantities are dimensionless)

Our derivation of FRGE is mostly standard, apart from:

- Non-standard ghost sector
- Appropriately chosen Type II cutoff, to eliminate certain singularities
- (Interpolated) spectral sums, rather than heat kernel

Ghost sector

- Same ghost sector as in [DB, New J. Phys. 14 (2012) 015005 [arXiv:1107.3110]]
- Basically (Q is the usual ghost operator)

$$\sqrt{\det Q^2} \quad \text{instead of} \quad \det Q$$

- Formally equivalent (in general there is multiplicative anomaly)
- Different FRG flows (beyond one-loop approximation)
- $\sqrt{\det Q^2}$ leads to exact cancellation on shell between ghost and pure-gauge dof
  $\Rightarrow$ on shell gauge-independence
- Checked: both versions lead to qualitatively similar results in the $\alpha = 0$ gauge

Cutoff

- Typical (Type I) rule: chose $\mathcal{R}_k$ such that

$$\Delta \to P_k\Big(\frac{\Delta}{k^2}\Big) \equiv \Delta + k^2 r_k\Big(\frac{\Delta}{k^2}\Big)\,.$$

- It can lead to singularities. For example, with optimized cutoff,

$$\frac{1}{2}\mathrm{Tr}\left[\frac{\partial_t \mathcal{R}_k}{\Delta - \frac{R}{d} + \mathcal{R}_k}\right] = \mathrm{Tr}\left[\frac{1}{1-\frac{\tilde{R}}{d}}\theta(k^2-\Delta)\right]$$

$\Rightarrow$ singularity at $\tilde{R} = d$

- Adopt (hybrid) Type II cutoff:

$$\begin{aligned}
\Delta_0 \equiv \Delta - \frac{R}{d-1} \quad &\to \quad P_k^{(0)}\Big(\frac{\Delta_0}{k^2}\Big) \equiv \Delta_0 + k^2 r_k\Big(\frac{\Delta_0}{k^2}\Big) \\
\Delta_1 \equiv \Delta - \frac{R}{d} \quad &\to \quad P_k^{(1)}\Big(\frac{\Delta_1}{k^2}\Big) \equiv \Delta_1 + k^2 r_k\Big(\frac{\Delta_1}{k^2}\Big) \\
\Delta_2 \equiv \Delta + \frac{2R}{d(d-1)} \quad &\to \quad P_k^{(2)}\Big(\frac{\Delta_2}{k^2}\Big) \equiv \Delta_2 + k^2 r_k\Big(\frac{\Delta_2}{k^2}\Big)
\end{aligned}$$

$\Rightarrow$ No explicit singularities in the functions being traced in the FRGE

Spectral sums

- We evaluated the traces directly as spectral sums:

$$\mathrm{Tr}W(\Delta_s) = \sum_n D_{n,s} W(\lambda_{n,s})$$

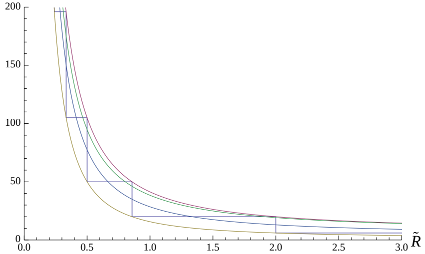
- Optimized cutoff:

  Pros: sums can be performed exactly

  Cons: result is a staircase function of $\tilde{R}$

- We opted to keep the advantage of working with analytical expressions, and we dealt with the cons using an interpolation:

  (similar to [Reuter, Weyer, 0804.1475])



Fixed-point equation

- At the end, we find the following fixed-point equation:

$$\tilde{f}'''(\tilde{R}) = \frac{\mathcal{N}(\tilde{f},\tilde{f}',\tilde{f}'';\tilde{R})}{\tilde{R}(\tilde{R}^4 - 54\tilde{R}^2 - 54)\Big((\tilde{R}-2)\tilde{f}'(\tilde{R}) - 2\tilde{f}(\tilde{R})\Big)}$$

  where $\mathcal{N}(\tilde{f},\tilde{f}',\tilde{f}'';\tilde{R})$ is a polynomial in all its variables.

- 3rd order equation $\Rightarrow$ 3 initial conditions
- Singularity at $\tilde{R}=0$ requires one analyticity condition that reduces number of independent initial conditions at origin to 2.

  (Note: something similar happens for scalar theory if we use $\rho = \phi^2$ as field [Comellas, Travesset])

  Note that singularity at $\tilde{R}=0$ is linked to order of the equation:

$$\partial_t \mathcal{R}_k \sim \dots + (\partial_t \tilde{f}_k''(\tilde{R}))\Big(\dots\Big) = \dots - 2\tilde{R}\tilde{f}_k'''(\tilde{R})\Big(\dots\Big) + \dots$$

- Singularity also at $\tilde{R}_\pm \simeq \pm 7.414$, originated by zero-mode of $h = g^{\mu\nu}h_{\mu\nu}$:

  $\Delta_0 \equiv \Delta - \frac{R}{d-1} \Rightarrow$ zero mode: $\tilde{\lambda}_0 = -\frac{\tilde{R}}{d-1} \Rightarrow$ at large $\tilde{R}$

  $\tilde{f}_k'''(\tilde{R})\sum_n D_n(1-\tilde{\lambda}_n^2)\theta(1-\tilde{\lambda}_n) = \tilde{f}_k'''(\tilde{R})D_0(1-\tilde{\lambda}_0^2)$ has a zero
- Non-linear equation $\Rightarrow$ also movable singularities

Small-$\tilde{R}$ expansion

- Expand in series at the origin:

$$\tilde{f}(\tilde{R}) = \sum_{n\geq 0} a_n \tilde{R}^n$$

- Plug into FP equation
  $\Rightarrow$ Coefficients a_n can be solved iteratively as function of a_0 and a_1

- Order-N truncation:
  Impose $a_{N+1} = a_{N+2} = 0$ and forget about higher terms

- Reproduce old results from polynomial truncations

- At $\tilde{R}_+$ there is a similar analiticity condition, hence also a series expansion with two free parameters, which can be treated in a similar way

Large-$\tilde{R}$ expansion – part 1

- At large $\tilde{R}$ solutions behave like

$$\tilde{f}(\tilde{R}) \sim A\tilde{R}^2 \left(1 + \sum_{n\geq 1} d_n \tilde{R}^{-n}\right)$$

  where $d_n = d_n(A)$ can be computed order by order
  $\Rightarrow$ only one free parameter in asymptotic expansion

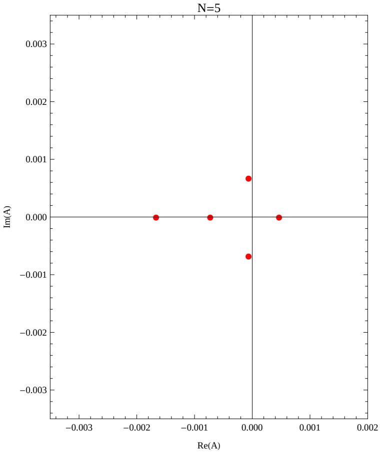
- Note for later: leading order is $\tilde{R}^2$

- Asymptotic expansion can be treated as a large-$\tilde{R}$ truncation, i.e. computing the beta functions for the couplings d_n and looking for fixed points

- In practice, we solve $d_n(A)$ for $n = 1, ...N + 1$, and then impose $d_{N+1}(A) = 0$

Large-$\tilde{R}$ expansion – part 2

- Fixed points in the complex A-plane



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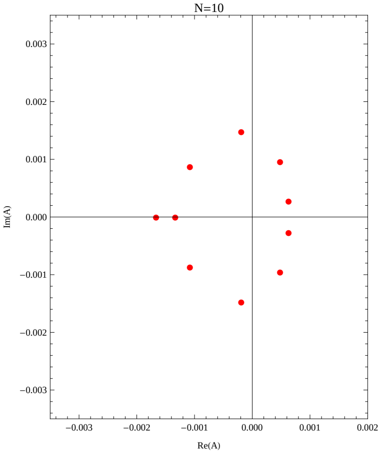

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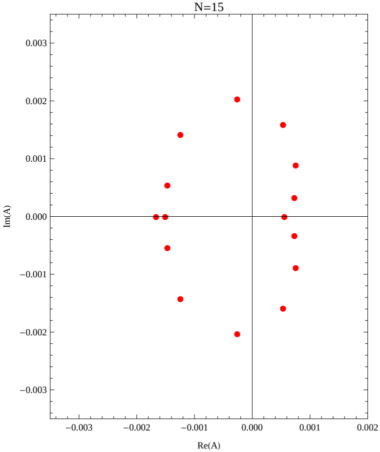

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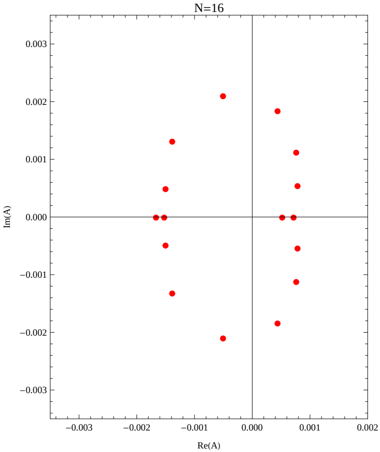

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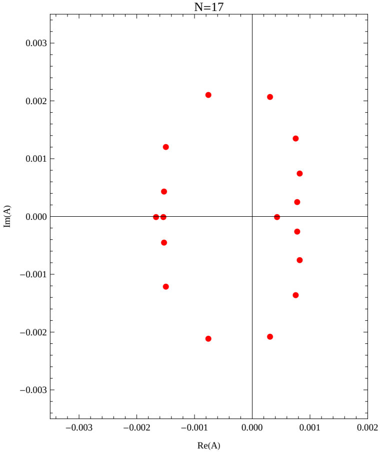

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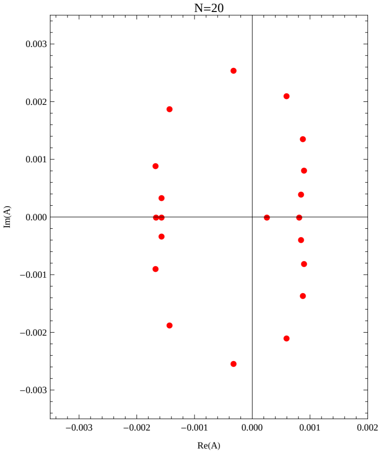

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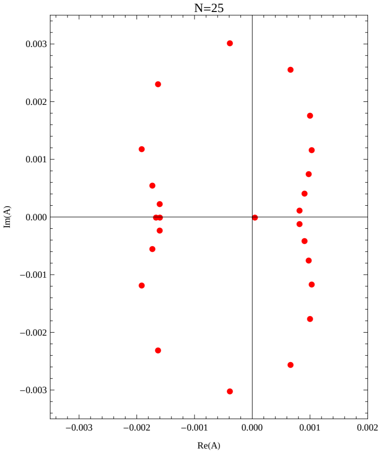

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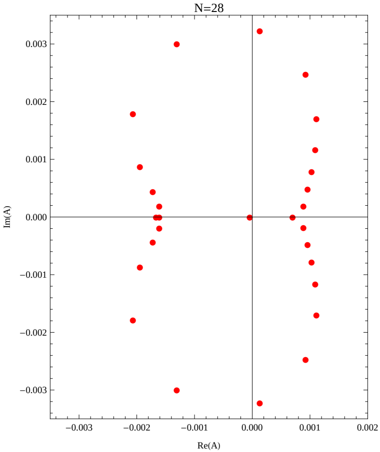

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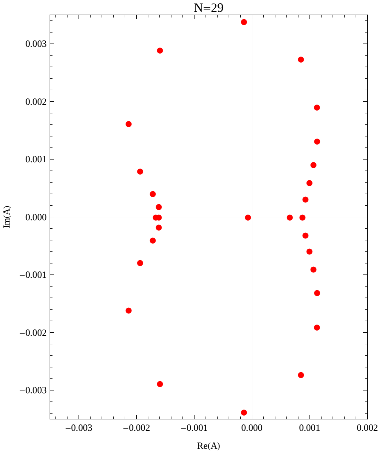

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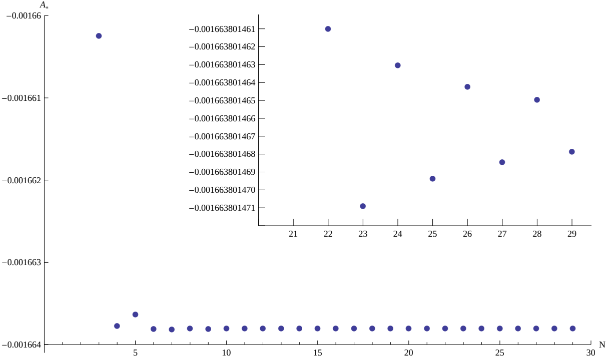
Large-$\tilde{R}$ expansion – part 3

- Fast convergence of FP on negative axis:



Large-$\tilde{R}$ expansion – part 3

- Fast convergence of FP on negative axis:

N	$10^3 A_1$	θ_0	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8
1	-1.319	-4.26	0.65							
2	-1.631	-1.64	9.19	29.2						
3	-1.660	-1.80	7.47	29.6	20.39					
4	-1.663	-1.82	7.25	29.4	17.3-4.38i	17.3+4.38i				
5	-1.663	-1.81	7.27	29.7	14.6-4.24i	14.6+4.24i	20.9			
6	-1.663	-1.82	7.26	29.5	14.0-3.19i	14.0+3.19i	19.4-4.20i	19.4+4.20i		
7	-1.663	-1.82	7.26	29.8	14.4-2.50i	14.4+2.50i	16.7-4.72i	16.7+4.72i	22.5	
8	-1.663	-1.82	7.26	29.5	14.8-3.55i	14.8+3.55i	15.8-2.32i	15.8+2.32i	21.6+3.83i	21.6-3.83i

$\Rightarrow$ only 1 irrelevant direction, increasing number of relevant directions

- Result compatible with [Machado,Saueressig], where single R^{-n} term was added, and found to be relevant

Numerical integration

- A better use for large-$\tilde{R}$ expansion should be to impose it as initial condition for a numerical integration of the FRGE

  (shooting backward from infinity)
- However the plot of singularities is much more complicated than in scalar case, and also fixed singularities are on the way
- In the range $-0.0035 \lesssim A \lesssim 0.0005$ the numerical integration can reach $\tilde{R}_+ + \epsilon$, and it can be shown (by scaling) that in the limit $\epsilon \to 0$ the analiticity condition at $\tilde{R}_+$ is satisfied in the whole range
- Combining numerical integration and series expansion at $\tilde{R}_+$, integration can be continued to $\tilde{R} < \tilde{R}_+$
- However, due to high sensitivity to order of the expansion and other effects, situation at $\tilde{R} = 0$ is not clear yet

  $\Rightarrow$ work in progress

Fixed-point action

- One generic conclusion can be drawn:

If a global fixed-point solution $\tilde{f}^*(\tilde{R})$ exists, then $\Gamma^* = \Gamma^*_{k=0} = A^* \int d^4x \sqrt{g} R^2$

$\Gamma^*_k = k^4 \int d^4x \sqrt{g} \tilde{f}^*(R/k^2)$ and limit $k \to 0$ corresponds to $\tilde{R} \to \infty$

- Resummation: polynomial truncations give non-trivial FP for $\tilde{R}^3$, $\tilde{R}^4$ and so on, but they must sum up to a function going like $\tilde{R}^2$ at infinity
- Agreement with [Bonanno [1203.1962]; Hindmarsh, Saltas [1203.3957]]
- Is it just dimensional analysis at work? (scale invariance $\to R^2$)

  In a sense yes: no anomalous scaling within $f(R)$ approximation ($\sim$ LPA)

  However: could R acquire an anomalous dimension beyond LPA?
  Does it make sense for $g_{\mu\nu}$ to acquire an anomalous dimension?
- The fact that FP theory is an R^2 theory requires more thinking about unitarity of full theory (where FP action could contain C^2)

Conclusions and outlook

Conclusions and outlook

What I have discussed:

- A suggestion for a different type of expansion, a near-MSS expansion, whose leading order is the $f(R)$-approximation ($\sim$LPA)
- We re-derived the FP equation for $f(R)$, eliminating some singularities, and discussed the role of the remaining singularities
- We studied the large-R truncations
  – Fast convergence, but unbounded action and many relevant directions
- We gave a general argument that the FP action is an R^2 action

Some open questions:

- Existence of global solutions of the differential FP equation
- Next order in near-MSS expansion: $f_2(R)C^2$. Anomalous dimension?
- Technical challenge: go to non-Einstein space and include derivatives of curvature (running of terms like $R\,F(-\nabla^2)R$)
- Can an asymptotically safe higher-derivative theory be unitary?