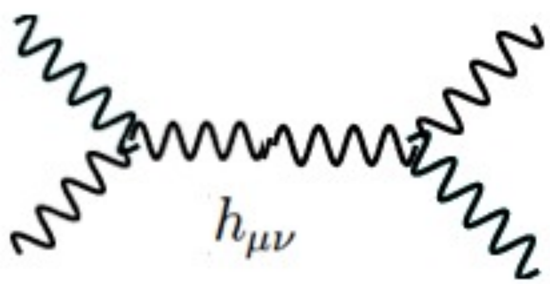
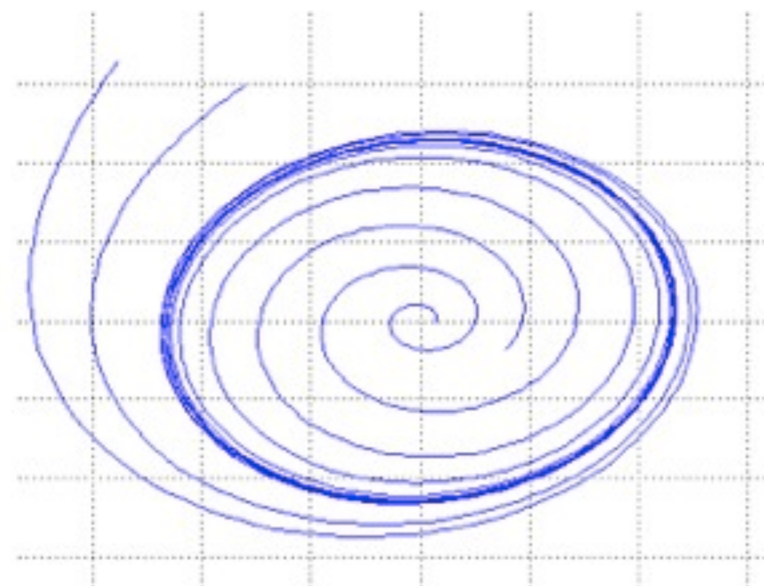
Limit Cycles and Quantum Gravity





Alejandro Satz - University of Maryland

International Seminar on Asymptotic Safety -- July 9th, 2012

Based on arXiv:1205.4218 (and forthcoming), in collaboration with Daniel Litim

Thanks to CONICET and NSF for support and the University of Sussex for hospitality

Outline

1. QG renormalization group
    1.1. Introduction
    1.2. Einstein-Hilbert truncation
    1.3. Conformal reduction

2. Minisuperspace truncation
    2.1. Reduction strategy
    2.2. The flow equations

3. Structure of the flow
    3.1. Fixed points and limit cycle
    3.2. Comparison of approximations

4. Emergence of limit cycle
    4.1. Tuning parameter
    4.2. Flow for increasing *n*
    4.3. Bifurcation point
    4.4. Flow for lower *n*

5. Summary and outlook
    5.1. Summary
    5.2. Perspectives for Asymptotic Safety
    5.3. Relevance for discrete gravity

Introduction

Quantized general relativity is perturbatively nonrenormalizable

Solutions →
- Treat theory as an effective low-energy description (Donoghue, ...) to be completed by new high energy physics (string theory, ...)
- Try to quantize the theory non-perturbatively
  - Canonically (LQG, mini/midi-superspace...)
  - Discretizing (spin foams, CDTs, Regge calculus...)
  - Following the Exact Renormalization Group

Legendre effective action Γ_k, defined by an IR cutoff term added to the action in the path integral.

$$\Delta_k S = \int \phi(p)\, \mathcal{R}_k(p^2)\, \phi(-p)$$

Leads to ERGE: $$k\,\partial_k\,\Gamma_k = \frac{1}{2}\,\mathrm{Tr}\left[\frac{k\,\partial_k\,R_k}{\Gamma_k^{(2)} + R_k}\right]$$

Asymptotically stable solutions →
- Fixed points
- Limit cycles

Einstein-Hilbert Truncation

For quantum gravity, Γ_k depends on a background metric $\bar{g}_{\mu\nu}$ and the expectation value of the fluctuation $\bar{h}_{\mu\nu}$. We consider only the case with dependence on the full metric $\bar{g}_{\mu\nu} + \bar{h}_{\mu\nu}$.

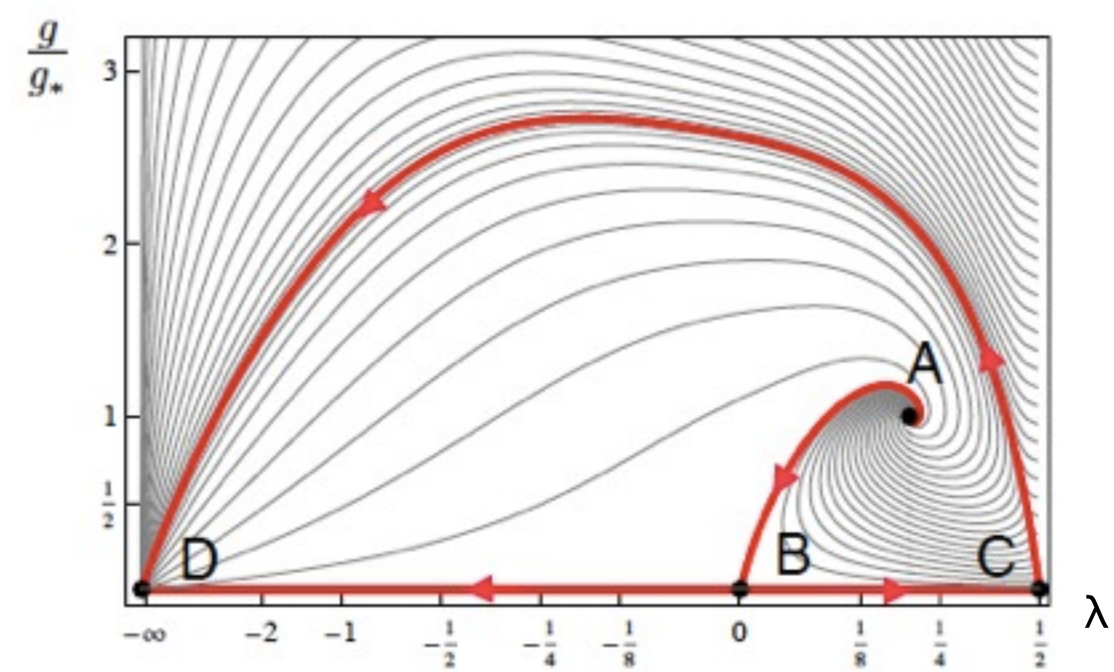
For the standard Einstein-Hilbert truncation we have the ansatz:

$$\Gamma_k = -\frac{1}{16\pi G_k} \int \sqrt{g}\,(R[g] - 2\Lambda_k)\Big|_{g=\bar{g}+\bar{h}} + S_{gf} + S_{gh}$$

Using the "optimized" cutoff and in a first-order approximation where the anomalous dimension is kept linear in g, the flow equations in four dimensions are:

$$\dot{g}_k = (2+\eta) g_k\,, \qquad \eta = -\frac{81 - 132\lambda_k + 100\lambda_k^2}{24\pi(1-2\lambda_k)^2}\, g_k$$

$$\dot{\lambda}_k = -(2-\eta)\lambda_k + \frac{g_k}{2\pi(1-2\lambda_k)}\left(1 + 4\lambda_k - \frac{\eta}{6}(5 - 4\lambda_k)\right)$$



A: UV non-Gaussian FP
B: Gaussian FP
C: Degenerate point
D: IR attractor

Conformal reduction

An approximation to the Einstein-Hilbert truncation, that preserves only the conformal factor of the metric as dynamical (Reuter-Weyer, ...).

$$g_{\mu\nu} = \chi^2(x)\hat{g}_{\mu\nu} \qquad \chi = \chi_B + f$$

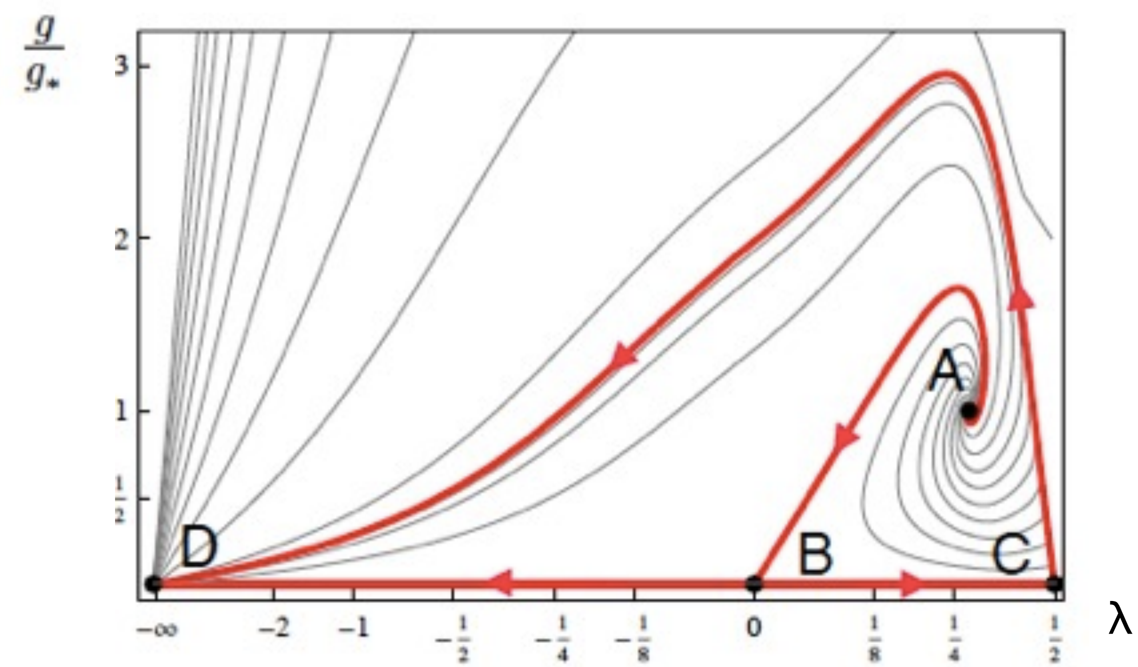
$$\Gamma_k[\bar{f};\chi_B] = -\frac{3}{8\pi G_k}\int d^4x\,\sqrt{\hat{g}}\left[-(\chi_B+\bar{f})\hat{\Box}(\chi_B+\bar{f}) + \frac{1}{6}\hat{R}(\chi_B+\bar{f})^2 - \frac{1}{3}\Lambda_k(\chi_B+\bar{f})^4\right]$$

$$\mathcal{R}_k[\chi_B] = -\frac{3}{4\pi G_k}\chi_B^2\,k^2\,R^{(0)}\left(\frac{-\hat{\Box}}{\chi_B^2 k^2}\right)$$

The flow equations are:

$$\dot{g}_k = (2+\eta)g_k\,, \qquad \eta = -\frac{2}{3\pi}\frac{g_k\,\lambda_k^2}{(1-2\lambda_k)^4}$$

$$\dot{\lambda}_k = (\eta-2)\lambda_k + \frac{g_k}{4\pi}\left(1-\frac{\eta}{6}\right)\frac{1}{1-2\lambda_k}$$



The main features of the full EH truncation are preserved in the conformal reduction.

Reduction strategy

Spatially flat FRW written in conformal form: $\mathrm{d}s^2 = a^2(t)\left[N^2(t)\,\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2)\right]$

Minisuperspace EH action: $S = -\frac{3v}{8\pi G}\int \mathrm{d}t\left[\frac{[a'(t)]^2}{N(t)} - \frac{\Lambda}{3}N(t)[a(t)]^4\right]$ *v* spatial volume parameter

We can choose to gauge-fix N=1. Then we have an example of the conformally reduced theory, with flat ĝ and with the conformal factor depending on a single coordinate instead of four.

Hence to derive the flow equations, we simply suppress the spatial dependence of the fluctuations on the traces that give the beta functions of the CR theory.

$$k\partial_k\Gamma_k[\bar{f};\chi_B] = \frac{1}{2}\mathrm{Tr}\left[\left(\Gamma_k^{(2)}[\bar{f};\chi_B] + \mathcal{R}_k[\chi_B]\right)^{-1} k\partial_k\mathcal{R}_k[\chi_B]\right]$$

$\longrightarrow$ $k\partial_k(G_k)^{-1} = \int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathcal{F}(p^2,k)$, simil. for Λ_k. $\mathcal{F}$ involves the cutoff function $\mathcal{R}^{(0)}\left(\frac{p^2}{k^2a_B^2}\right)$

Introduce a δ-function to supress the dependence on p_i, i = 1,2,3.

$$k\partial_k(G_k)^{-1} = \int\frac{\mathrm{d}^4p}{(2\pi)^4}\delta^3\left(\frac{p_i}{a_Bk}\right)\mathcal{F}(p^2,k)$$, simil. for Λ_k .

The flow equations

Note that we could include an arbitrary constant *c* in the δ-function.

Reason for this freedom: arbitrariness of the spatial volume *v*

→ multiplicative ambiguity in the definition of G for a minisuperspace theory with flat reference metric.

We are led to the flow equations:

$$\dot{g}_k = (2+\eta)g_k\,, \qquad \eta = -\frac{64c}{3}\frac{g_k\,\lambda_k^2}{(1-2\lambda_k)^4}$$

$$\dot{\lambda}_k = (\eta-2)\lambda_k + 8\,c\,g_k\left(1-\frac{\eta}{3}\right)\frac{1}{1-2\lambda_k}$$

There is an alternative derivation of these equations starting directly from an ansatz for a minisuperspace Γ_k instead of the full CR theory. It requires a hand-picked choice for the flow of *v*. (Cf. Manrique, Rechenberger and Saueressig on Lorentzian RG.)

Details in Litim and Satz, forthcoming.

The flow equations (cont.)

None of the essential features of the flow depend on the choice of *c*. We choose a value that maximizes the similarity with the full CR theory:

$$\dot{g}_k = (2+\eta)g_k\,, \qquad \eta = -\frac{2}{3\pi}\frac{g_k\,\lambda_k^2}{(1-2\lambda_k)^4}$$

$$\dot{\lambda}_k = (\eta-2)\lambda_k + \frac{g_k}{4\pi}\left(1-\frac{\eta}{3}\right)\frac{1}{1-2\lambda_k}$$

This is the system we will study in the rest of the talk.

The flow equations (cont.)

None of the essential features of the flow depend on the choice of *c*. We choose a value that maximizes the similarity with the full CR theory:

$$\dot{g}_k = (2+\eta)g_k\,, \qquad \eta = -\frac{2}{3\pi}\frac{g_k\,\lambda_k^2}{(1-2\lambda_k)^4}$$

$$\dot{\lambda}_k = (\eta-2)\lambda_k + \frac{g_k}{4\pi}\left(1-\frac{\eta}{3}\right)\frac{1}{1-2\lambda_k}$$

Was a 6 in full CR theory. General value for δ^{4-n} suppression would be n+2.

This is the system we will study in the rest of the talk.

The flow equations (cont.)

None of the essential features of the flow depend on the choice of *c*. We choose a value that maximizes the similarity with the full CR theory:

$$\dot{g}_k = (2+\eta)g_k\,, \qquad \eta = -\frac{2}{3\pi}\frac{g_k\,\lambda_k^2}{(1-2\lambda_k)^4}$$

$$\dot{\lambda}_k = (\eta-2)\lambda_k + \frac{g_k}{4\pi}\left(1-\frac{\eta}{3}\right)\frac{1}{1-2\lambda_k}$$

→ Was a 6 in full CR theory. General value for $\delta^{4\text{-}n}$ suppression would be n+2.

This is the system we will study in the rest of the talk.

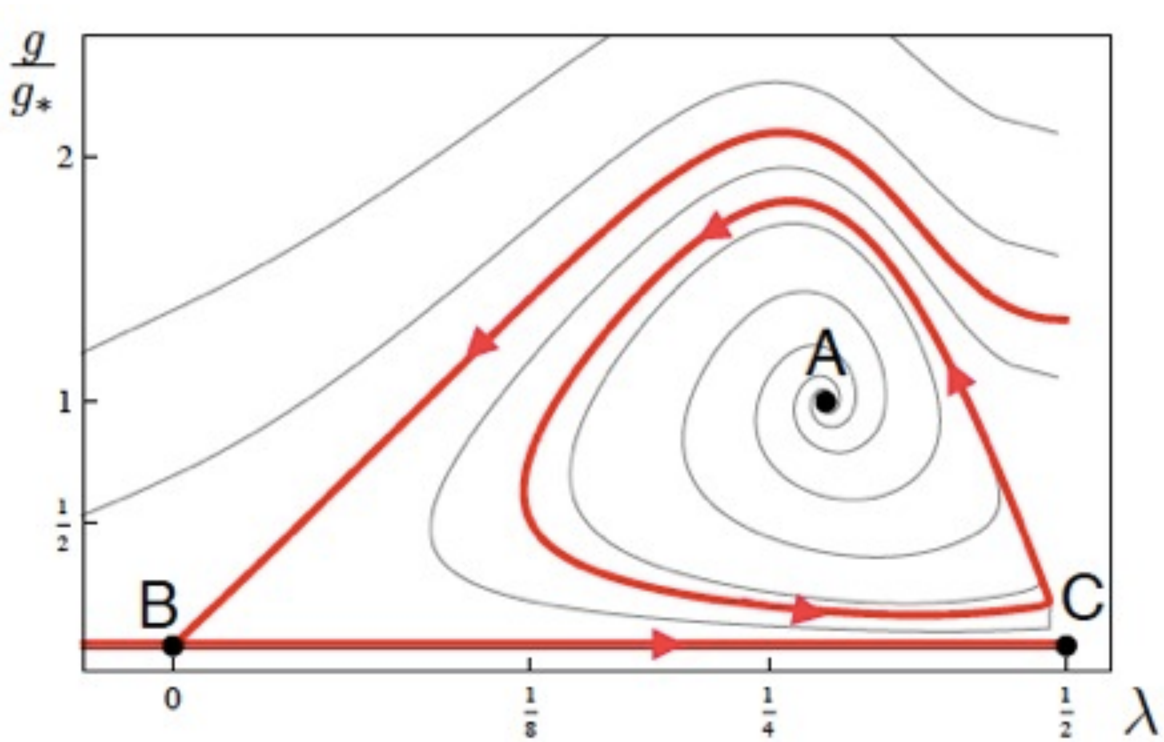
Suppressing the spatial fluctuations leads to

→ Very similar flow equations

→ Many qualitative similarities in the flow.

→ But also important dissimilarities!

Fixed points and limit cycle



A: Non-Gaussian fixed point: $g^* = 3.683$

$\lambda^* = 0.283$

Critical exponents θ^* are complex, with $\mathrm{Re}(\theta^*) = 1.77 > 0$

→ UV-attractive, similar to the full theory.

B: Gaussian fixed point:

$g^* = 0 = \lambda^*$, critical exponents $\theta^* = \pm 2$.

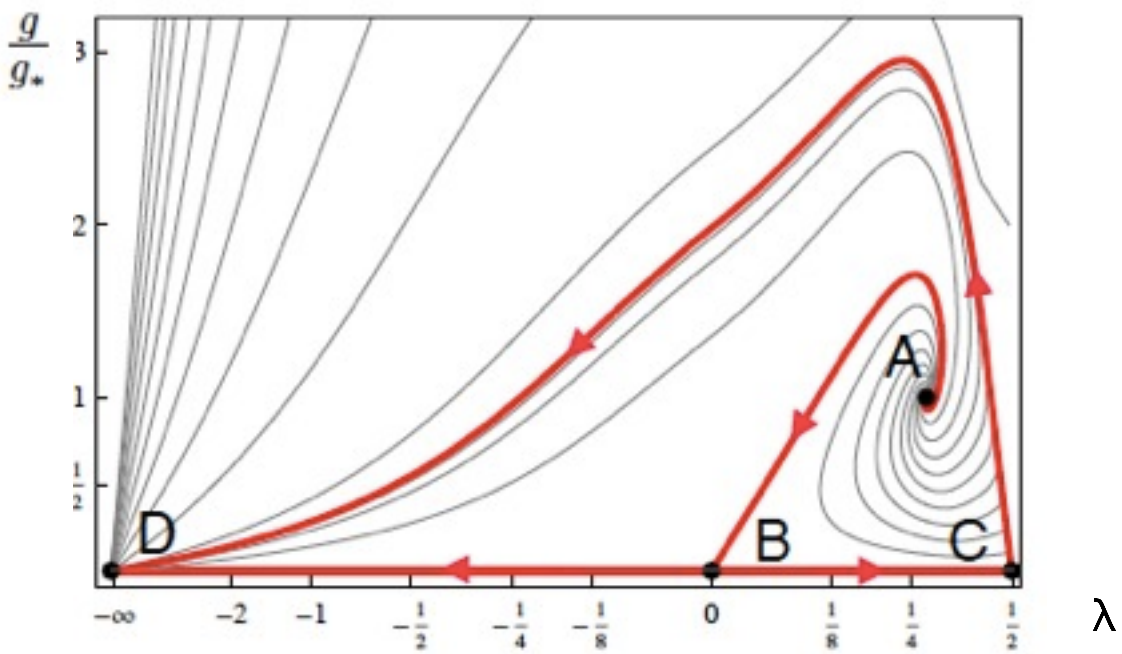
C: Degenerate fixed point at $g = 0$, $\lambda = 1/2$.

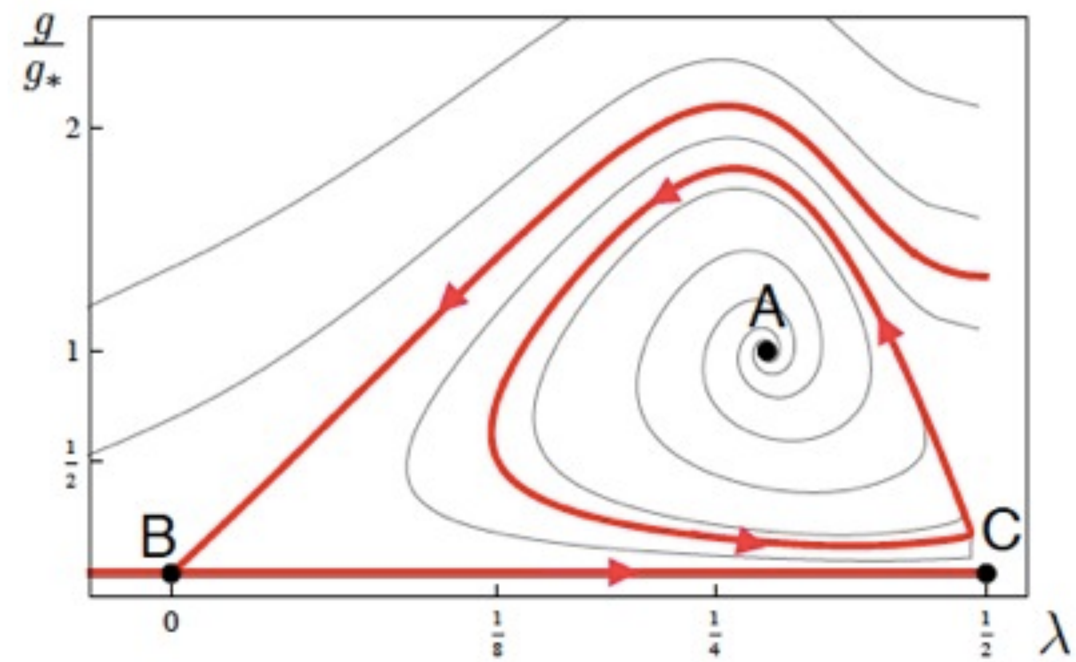
Also present, IR-attractive fixed point at $g = 0$, $\lambda \to -\infty$.

Qualitative new feature: limit cycle separating the NGFP from the semiclassical regime.

Limit cycle is IR-attractive from both inside and outside.

Comparison of approximations





- Two families of trajectories:
  1) From NGFP (A)
  2) From $\lambda = 1/2$ singularity.

Both IR-running towards D attractor.

- Two separatrices
  1) From NGFP (A) to GFP (B).
  2) from C to D.

- Extended semiclassical regime for particular trajectories leaving A.

- Three families of trajectories:
  1) From NGFP (A) to limit cycle
  2) From $\lambda = 1/2$ singularity to limit cycle
  3) From $\lambda = 1/2$ singularity to D

- Two separatrices
  1) Limit cycle.
  2) From $\lambda = 1/2$ singularity to GFP (B)

- Extended semiclassical regime for particular trajectories leaving $\lambda = 1/2$.

Tuning parameter

We can study the emergence of the limit cycle introducing a parameter n, representing the number of dimensions on which the conformal factor fluctuates.

$$k\partial_k (G_k)^{-1} = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \delta^{(4-n)}\left(\frac{p_i}{a_B k}\right) \mathcal{F}(p^2, k) \quad ,$$

simil. for Λ_k

→

$$\dot{g}_k = (2+\eta) g_k \, , \qquad \eta = -\frac{2}{3\pi} \frac{g_k \lambda_k^2}{(1-2\lambda_k)^4}$$

$$\dot{\lambda}_k = (\eta - 2)\lambda_k + \frac{g_k}{4\pi}\left(1 - \frac{\eta}{n+2}\right)\frac{1}{1-2\lambda_k}$$

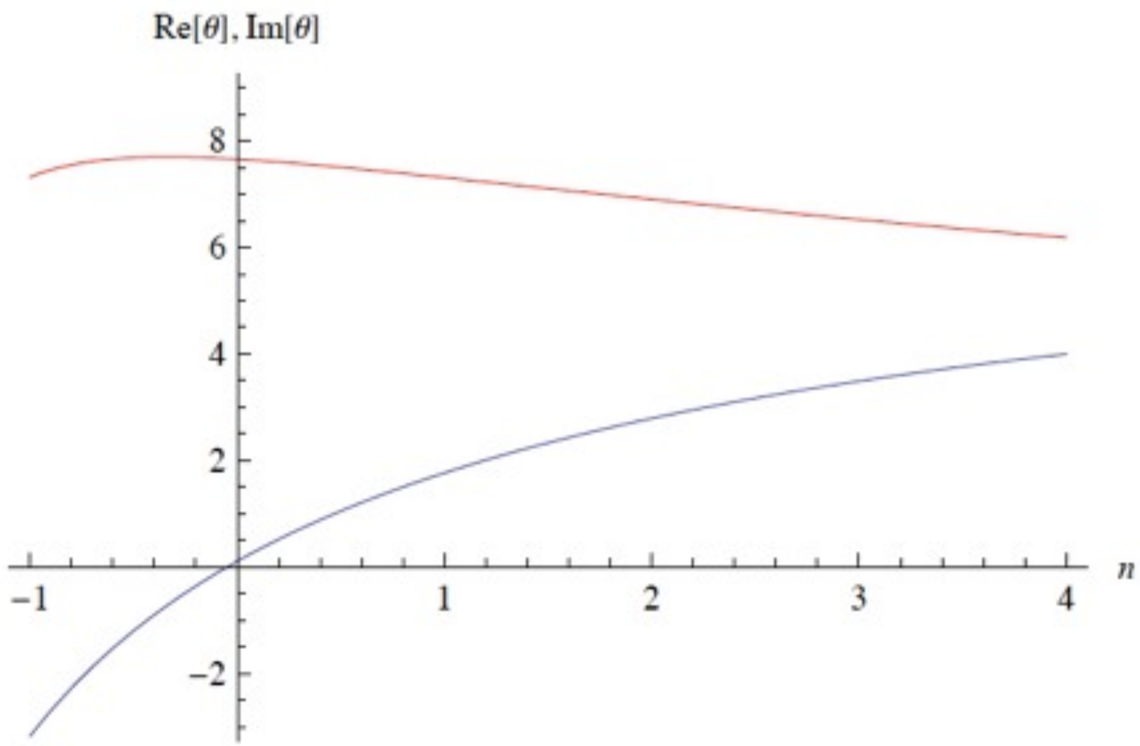
(with an *n*-dependent choice of *c*)

n = 1: minisuperspace approximation
n = 4: full conformally reduced theory

The four fixed points A, B, C, D exist for all values of *n* > 0. The location of the NGFP is almost insensitive to *n*.

n	g_k^*	λ_k^*
1	3.683	0.287
2	4.113	0.283
3	4.420	0.281
4	4.640	0.279

Flow for increasing *n*

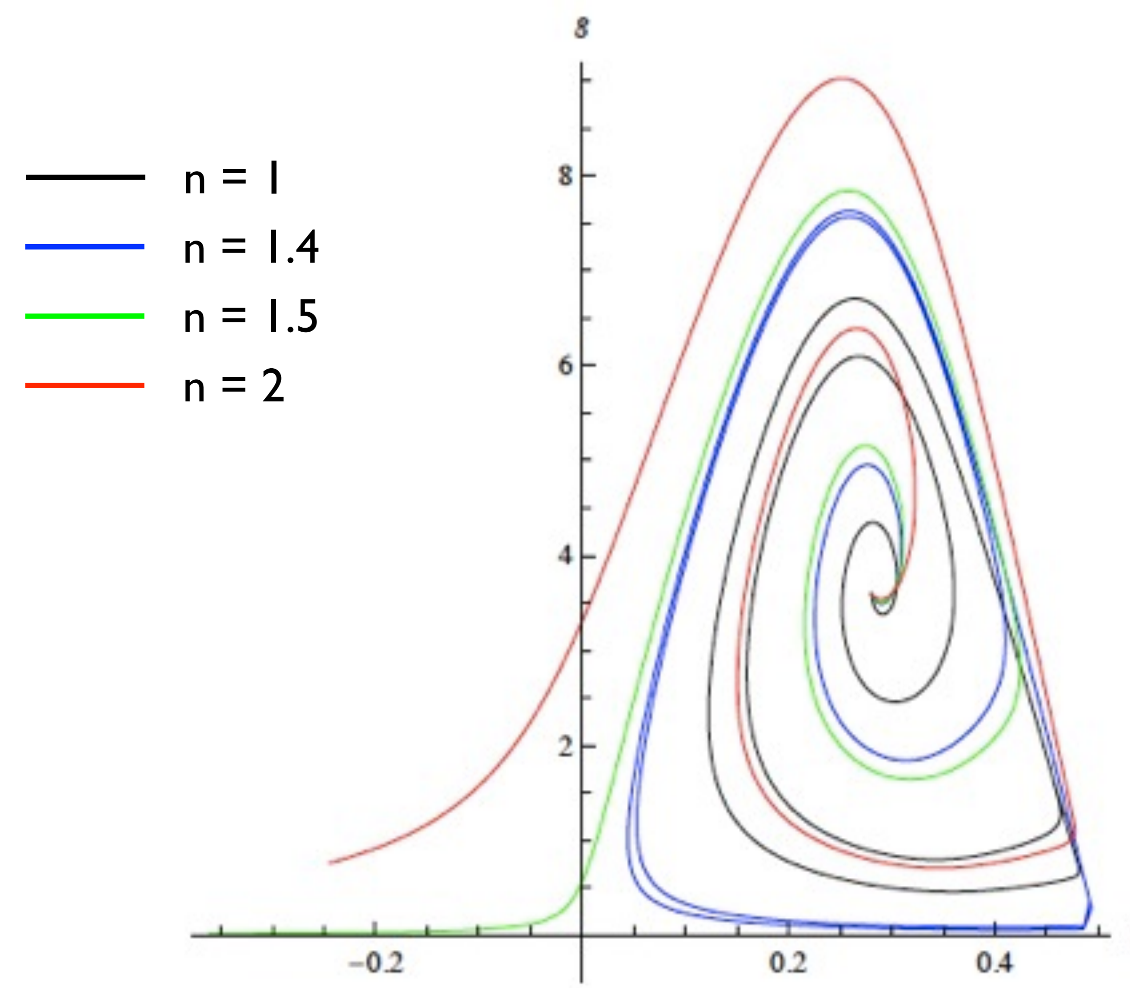


Re(θ^*) increases with *n*, showing that the fixed point is more strongly IR-repulsive as *n* grows.

Consequently, the size of the limit cycle surrounding the NGFP increases with *n*.

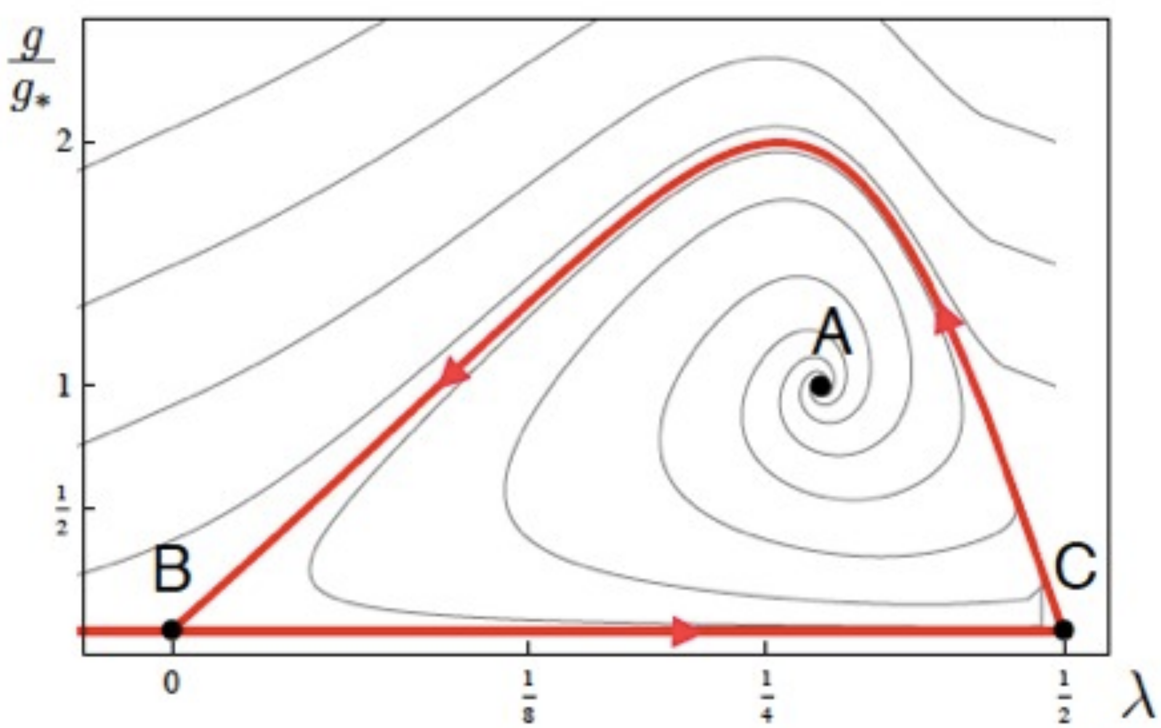
However, upon increasing *n* a point is eventually reached where the limit cycles disappears.

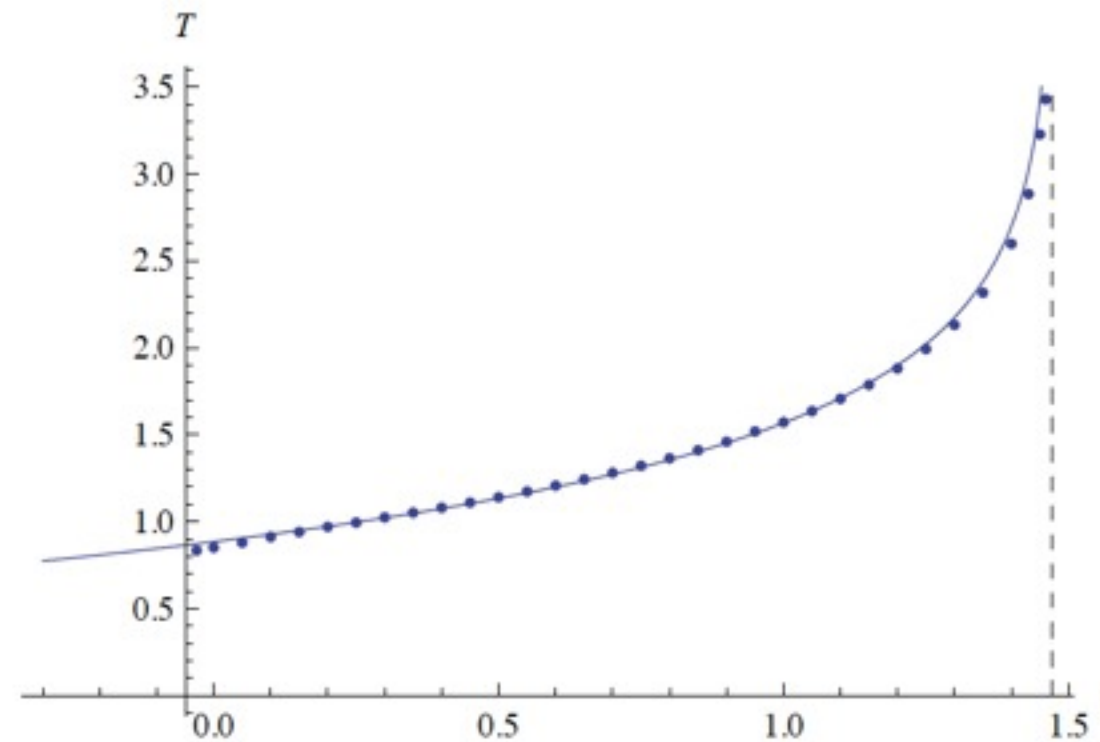
After that, trajectories from the NGFP escape freely towards the IR attractor D.



Bifurcation point

At the critical value $n^* \approx 1.4715$, the limit cycle collides with the Gaussian fixed point B and the degenerate fixed point C. The periodicity of the limit cycle in t diverges logarithmically as $n \to n^*$ from below.





At $n = n^*$ there are only two families of trajectories: those that flow out from the UV fixed point A into the degenerate limit cycle, and those that flow from the singularity at $\lambda = 1/2$ to the D attractor at $\lambda \to -\infty$.

All trajectories from A exhibit an extended semiclassical regime (no fine tuning!)

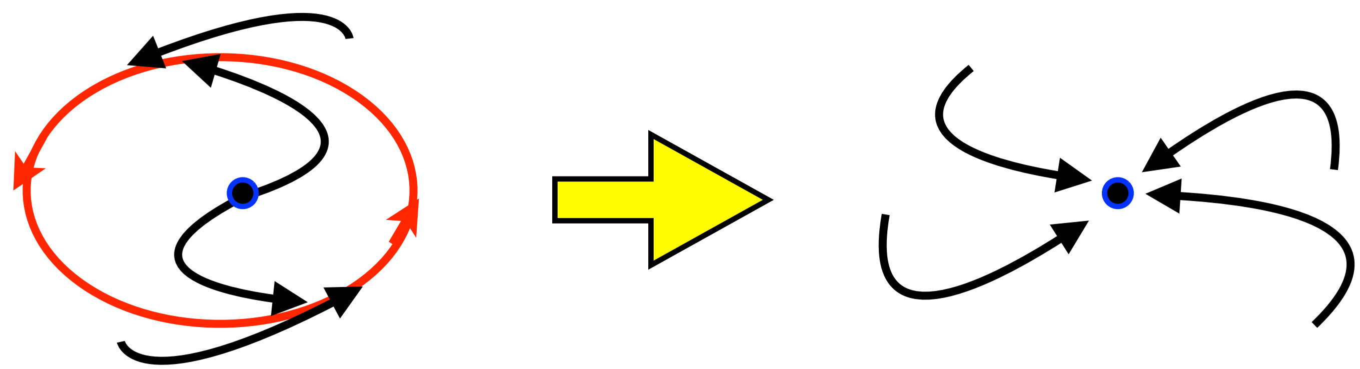
For $n > n^*$ the limit cycle disappears and the flow resembles qualitatively the one of the full theory.

Flow for lower n

For $n < 1$, the size of the limit cycle keeps decreasing as $\mathrm{Re}(\theta^*)$ decreases and the NGFP becomes less strongly IR-repulsive.

At $n \approx -0.05$, the limit cycle shrinks to a point and vanishes. $\mathrm{Re}(\theta^*)$ becomes negative, and the NGFP becomes IR-attractive.

Hopf bifurcation pattern



n does not have a physical interpretation in this regime.

Summary

- The RG flow of quantum gravity is studied in a minisuperspace approximation.
  - The truncation is derived from the Einstein-Hilbert theory for the conformal factor by suppressing fluctuations in 3 (more generally, in *4-n*) dimensions.
  - The flow equations generalize those of the $n = 4$ theory.

- In addition to the well known Gaussian non-Gaussian fixed points, the theory contains a limit cycle.
  - Trajectories run into the limit cycle when flowing to the IR both from the NGFP and from the semiclassical region.

- The size and period of the limit cycle grow with n until a bifurcation event at $n^* \approx 1.4715$.
  - At n^*, all trajectories leaving the NGFP exhibit an extended semiclassical regime and run into the (degenerate) limit cycle. Above n^* there is no limit cycle and the flow resembles the one for the full theory.
  - For $n < n^*$ the limit cycle decreases until vanishing in (at $n \approx -0.05$) a Hopf bifurcation.

Perspectives for Asymptotic Safety

The limit cycle is found in an ultra-simplified model, and is not present when including the full spatial dependence of the conformal factor or the spin 2 fluctuations.

However, the simplicity and naturalness of the *n*-tweaking that creates the limit cycle suggests that similar behaviour might be also present in a more rich and physical truncation (e.g., involving nonlocal operators?)

We must bear in mind the possibility of a limit cycle (instead of a fixed point) as the UV limit of the theory when followed upwards from the IR regime, and understand better its physical implications

The bifurcation point scenario offers a possibility of obtaining naturally a good semiclassical limit without fine-tuning the initial conditions for the RG flow.

Relevance for discrete gravity

In Euclidean/Causal Dynamical Triangulations, the spectral dimension n_s flows from 4 in the IR to respectively n_E =1.457 and n_C =1.80 in the UV.

The value n_E is intriguingly close to our bifurcation point $n^* \approx 1.4715$.

Assume the EDT can be understood as the microscopic theory descibed effectively, in some approximation, by our model with an effective value of n.

- $n_E > n^*$ implies that the effective theory reaches a UV fixed point.
- $n_E < n^*$ implies that the effective theory contains a limit cycle and no UV fixed point is reached.