THE RENORMALIZATION GROUP AND WEYL-INVARIANCE

Asymptotic Safety Seminar Series 2012

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Outline

- Weyl-invariance & Weyl geometry
- Free Matter in external gravity
- Effective Average Action (EAA)
- Interacting matter
- Dynamical gravity

Weyl-invariance

- Scale transformation $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$

maps one theory to another

- Scaling dimension defined through:

$$S(g_{\mu\nu}, \psi_a, g_i) = S(\Omega^2 g_{\mu\nu}, \Omega^{w_a}\psi_a, \Omega^{w_i} g_i)$$

"weights"

- Weyl transformation $\Omega = \Omega(x)$

- Introduce a dilaton as a *local mass scale*: $\mu \to \chi(x)$ $\chi \mapsto \Omega^{-1}\chi$

- One can construct a (nonmetric) connection and a Weyl-covariant derivative:

$$\hat{\Gamma}_{\mu}{}^{\lambda}{}_{\nu} = \Gamma_{\mu}{}^{\lambda}{}_{\nu} - \delta^{\lambda}_{\mu} b_{\nu} - \delta^{\lambda}_{\nu} b_{\mu} + g_{\mu\nu} b^{\lambda}$$

pure gauge field

$$D_\mu t = \hat{\nabla}_\mu t - w b_\mu t$$

$$b_\mu = -\chi^{-1}\partial_\mu \chi$$

- This defines the (integrable) **Weyl geometry** on the manifold.
- Curvatures are then built as usual:

$$[D_\mu, D_\nu]v^\rho = \mathcal{R}_{\mu\nu}{}^\rho{}_\sigma v^\sigma$$

- If every parameter in the theory is rewritten as a dilaton coupling

$$g_i = \chi^{-w_i}\hat{g}_i$$

and all derivatives and curvatures are replaced by Weyl-covariant derivatives and curvatures, we obtain a Weyl-invariant action (**Stückelberg trick**).

- The final recipe amounts to:

$$\hat{S}(g_{\mu\nu}, \chi, \psi_a, \hat{g}_i) = S(\chi^2 g_{\mu\nu}, \chi^{w_a}\psi_a, \chi^{w_i} g_i)$$

Free Matter - Standard Measure

Conformally coupled scalar field in external gravitational field

- Metric in field space:

$$\mathcal{G}(\phi, \phi') = \mu^2 \int dx \sqrt{g}\, \phi\, \phi'$$

- Action:

$$S_S(g_{\mu\nu}, \phi) = \frac{1}{2}\mathcal{G}\left(\phi, \frac{\Delta^{(0)}}{\mu^2}\phi\right) \qquad \Delta^{(0)} = -\nabla^2 + \frac{d-2}{4(d-1)}R$$

- Weyl covariance of the Laplacian means:

$$\Delta^{(0)}_{\Omega^2 g} = \Omega^{-1-\frac{d}{2}}\Delta^{(0)}_g \Omega^{\frac{d}{2}-1} \quad \xrightarrow{\Omega = 1+\omega} \quad \delta_\omega \Delta^{(0)} = -2\omega\Delta^{(0)}$$

- From the gaussian integral we obtain the Effective Action (EA):

$$\Gamma^I(\phi, g_{\mu\nu}) = S_S(\phi, g_{\mu\nu}) + \frac{1}{2}\mathrm{Tr}\log\left(\frac{\Delta^{(0)}}{\mu^2}\right)$$

different for fermion field

$$\Delta^{(1/2)} = -\nabla^2 + \frac{R}{4}$$

- A Maxwell field gives a contribution:

$$S_M(A_\mu, g_{\mu\nu}) + \frac{1}{2}\mathrm{Tr}\log\left(\frac{\Delta^{(1)}}{\mu^2}\right) - \mathrm{Tr}\log\left(\frac{\Delta^{(gh)}}{\mu^2}\right)$$

- These two operators are not Weyl-covariant (*d=4*):

$$\delta_\omega \Delta^{(gh)} = -2\omega\Delta^{(gh)} \underbrace{- 2\nabla^\nu\omega\nabla_\nu}_{\rho^{(gh)}}$$

$$\delta_\omega \Delta^{(1)\,\nu}_{\mu} = -2\omega\Delta^{(1)\,\nu}_{\mu} \underbrace{+ 2\nabla_\mu\omega\nabla^\nu - 2\nabla^\nu\omega\nabla_\mu - 2\nabla_\mu\nabla^\nu\omega}_{\rho^{(1)}}$$

The Trace Anomaly

- Under an infinitesimal Weyl transformation:

$$\delta_\omega \Gamma^I = \int dx \, \frac{\delta \Gamma^I}{\delta g_{\mu\nu}} 2\omega g_{\mu\nu} = -\int dx \sqrt{g}\, \omega \langle T^\mu_\mu \rangle$$

- From the proper-time representation of the EA for a Weyl-covariant operator, and the Heat Kernel expansion, one easily derives the relation:

$$\delta_\omega \Gamma^I = -\frac{1}{(4\pi)^{d/2}} \int dx \sqrt{g}\, \omega \, b_d(\Delta)$$

- The coefficients can be computed in dimension 2 and 4 giving:

$$\langle T^\mu{}_\mu \rangle = \frac{c}{24\pi} R$$

$$c = n_S + n_D$$

2D

$$\langle T^\mu{}_\mu \rangle = c\, C^2 - aE$$

$$a = \frac{1}{360(4\pi)^2} (n_S + 11 n_D + 62 n_M)$$

$$c = \frac{1}{120(4\pi)^2} (n_S + 6 n_D + 12 n_M)$$

4D

- The trace anomaly is a manifestation of the dependence on the mass scale:

$$\int dx\sqrt{g}\,\langle T^{\mu}_{\mu}\rangle = -\mu\frac{d}{d\mu}\frac{1}{2}\mathrm{Tr}\log\frac{\Delta}{\mu^2}$$

- In the spin 1 case we have Weyl noncovariant operators, giving an additional contribution

$$\frac{1}{2}\mathrm{Tr}\rho^{(1)}e^{-t\Delta^{(1)}} - \mathrm{Tr}\rho^{(gh)}e^{-t\Delta^{(gh)}}$$

- One can see that these contributions cancel in the traces, giving the correct result

$$\delta_\omega\Gamma^I = \frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\left[b_4(\Delta^{(1)}) - 2b_4(\Delta^{(gh)})\right]$$

Free Matter - New Measure

- One can construct Weyl-invariant metrics in field space replacing the mass scale with the dilaton:

$$\mathcal{G}_S(\phi,\phi') = \int d^4x\sqrt{g}\,\chi^2\phi\phi' \ ,$$

$$\mathcal{G}_D(\bar{\psi},\psi') = \int d^4x\sqrt{g}\,\frac{1}{2}\chi[\bar{\psi}\psi' + \bar{\psi}'\psi] \ ,$$

$$\mathcal{G}_M(A,A') = \int d^4x\sqrt{g}\,\chi^2 A_\mu g^{\mu\nu} A'_\nu$$

- Likewise define covariant operators:

$$\mathcal{O}_S = \chi^{-2}\Delta^{(0)} \ ,$$

$$\mathcal{O}_D = \chi^{-2}\Delta^{(1/2)} \ ,$$

$$\mathcal{O}_{M\mu}{}^{\nu} = \chi^{-2} g_{\mu\sigma}\left(\Delta^{(1)}\right)^{\sigma\nu} \ ,$$

$$\mathcal{O}_{gh} = \chi^{-2}\Delta^{(gh)}$$

- This will also define a new EA:

$$\Gamma^{II} = S + \frac{1}{2}\mathrm{Tr}\log\mathcal{O}$$

- While the standard EA is anomalous,

$$\delta_\omega \Gamma^{I} \neq 0$$

the new one is Weyl-invariant by construction:

$$\delta_\omega \Gamma^{II} = \int dx \sqrt{g}\omega \left(2\frac{\delta \Gamma^{II}}{\delta g_{\mu\nu}} g_{\mu\nu} - \frac{\delta \Gamma^{II}}{\delta \chi}\chi \right) = 0$$

Wess-Zumino Action

- The Wess-Zumino action is defined through the variation of the (standard) EA under a finite Weyl transformation:

$$\Gamma^I(g^\Omega) - \Gamma^I(g) = \Gamma_{WZ}(g, \mu\Omega)$$

- Important because all the dilaton-dependence in the Weyl-invariant EA comes from the Wess-Zumino term:

$$\Gamma^{II}(g,\chi) = \Gamma^I(g) + \Gamma_{WZ}(g,\chi)$$

- Also, from the relation

$$\int dx \sqrt{g}\, \frac{\delta \Gamma_{WZ}}{\delta g_{\mu\nu}}\bigg|_{(g,\chi=\mu)} 2\omega g_{\mu\nu} = 0$$

we see that the Weyl-invariant EA reproduces the trace anomaly in the gauge in which the dilaton is constant

Wess-Zumino action from integration of the anomaly

One-parameter family of Weyl transformations:

$$\Omega(t) : [0,1] \mapsto [1,\Omega] \qquad g(t)_{\mu\nu} = g_{\mu\nu}^{\Omega(t)}$$

$$\Gamma_{WZ}(g_{\mu\nu},\Omega) = \int_0^1 dt \int dx \, \frac{\delta\Gamma}{\delta g_{\mu\nu}}\bigg|_{g(t)} \delta g(t)_{\mu\nu} = -\int_0^1 dt \int dx \, \sqrt{g(t)} \langle T^\mu_\mu \rangle_k \Omega(t)^{-1} \frac{d\Omega}{dt}$$

For instance:

2D

$$\Gamma_{WZ}(g_{\mu\nu},\mu e^\sigma) = -\frac{c}{24\pi}\int d^2x \sqrt{g}\left(R\sigma - \sigma\nabla^2\sigma\right)$$

$$\Gamma_{WZ}(g_{\mu\nu},\mu e^\sigma) = -\int d^4x \, \sqrt{g} \left\{ c\, C^2\sigma - a\left[\left(E - \frac{2}{3}\Box R\right)\sigma + 2\sigma\Delta_4\sigma\right]\right\}$$

$$\Delta_4 = \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}R\Box + \frac{1}{3}\nabla^\mu R\nabla_\mu$$

4D

Effective Average Action

- Meaning of the cutoff k in the Weyl-invariant approach:

> *The cutoff must be allowed to be a generic non-negative function on spacetime. We will take it to be proportional to the dilaton.*

- Cutoff term:

k^2/χ^2

eigenvalues of the Weyl covariant operator $\mathcal{O}$

$$\Delta S_k(g_{\mu\nu}, \phi) = \frac{1}{2} u^2 \sum_n a_n^2 r\left(\tilde{\lambda}_n\right)$$

The EAA:

$$k\frac{d\Gamma_k}{dk} = \frac{1}{2}\mathrm{Tr}\left[\frac{\delta^2(\Gamma_k + \Delta S_k)}{\delta\phi\delta\phi}\right]^{-1} k\frac{d}{dk}\frac{\delta^2\Delta S_k}{\delta\phi\delta\phi}$$

is in the two cases:

$$R_k(\Delta) = k^2 r(\Delta/k^2)$$

$$k\frac{d\Gamma_k^I}{dk} = \mathrm{Tr}\frac{r(\Delta/k^2) - (\Delta/k^2)r'(\Delta/k^2)}{(\Delta/k^2) + r(\Delta/k^2)}$$

k is a scale

$$u\frac{d\Gamma_u^{II}}{du} = \mathrm{Tr}\frac{r(\mathcal{O}/u^2) - (\mathcal{O}/u^2)r'(\mathcal{O}/u^2)}{(\mathcal{O}/u^2) + r(\mathcal{O}/u^2)}$$

k is a field

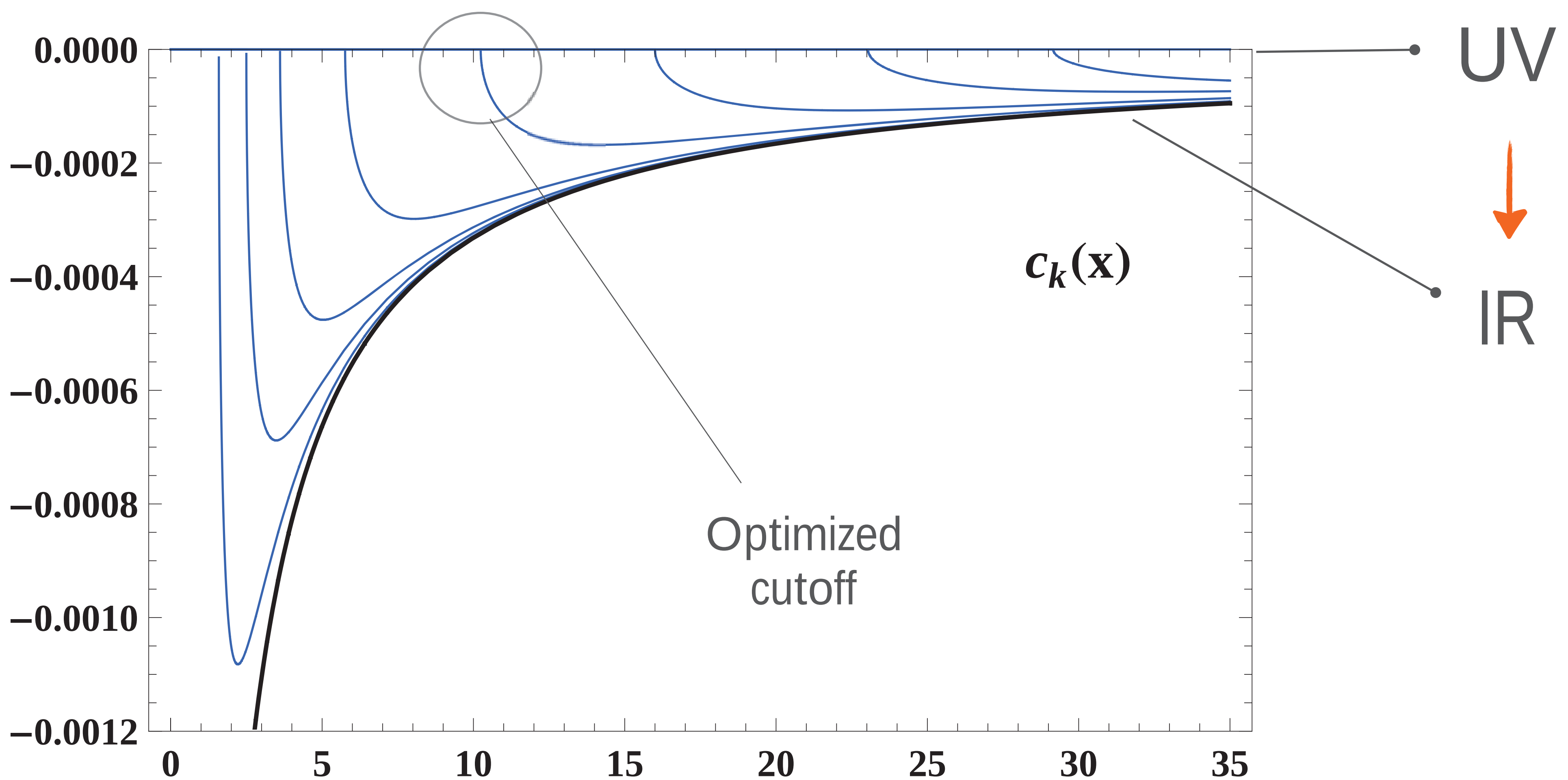
The c-anomaly in d=2

- General curvature-squared truncation

nonlocal form factor

$$\Gamma_k^I = \int d^2x \sqrt{g} \left[a_k + b_k R + R\, c_k(\Delta) R \right] + O\left(R^3\right)$$

- Beta function of c has been computed in [Codello, *Ann.Phys.* **325** (2010) 1727]



- When the form factor admits a series expansion: $c_k(\Delta) = \frac{1}{k^2} \sum_{n=1}^{\infty} c_n \frac{k^{2n}}{\Delta^n}$

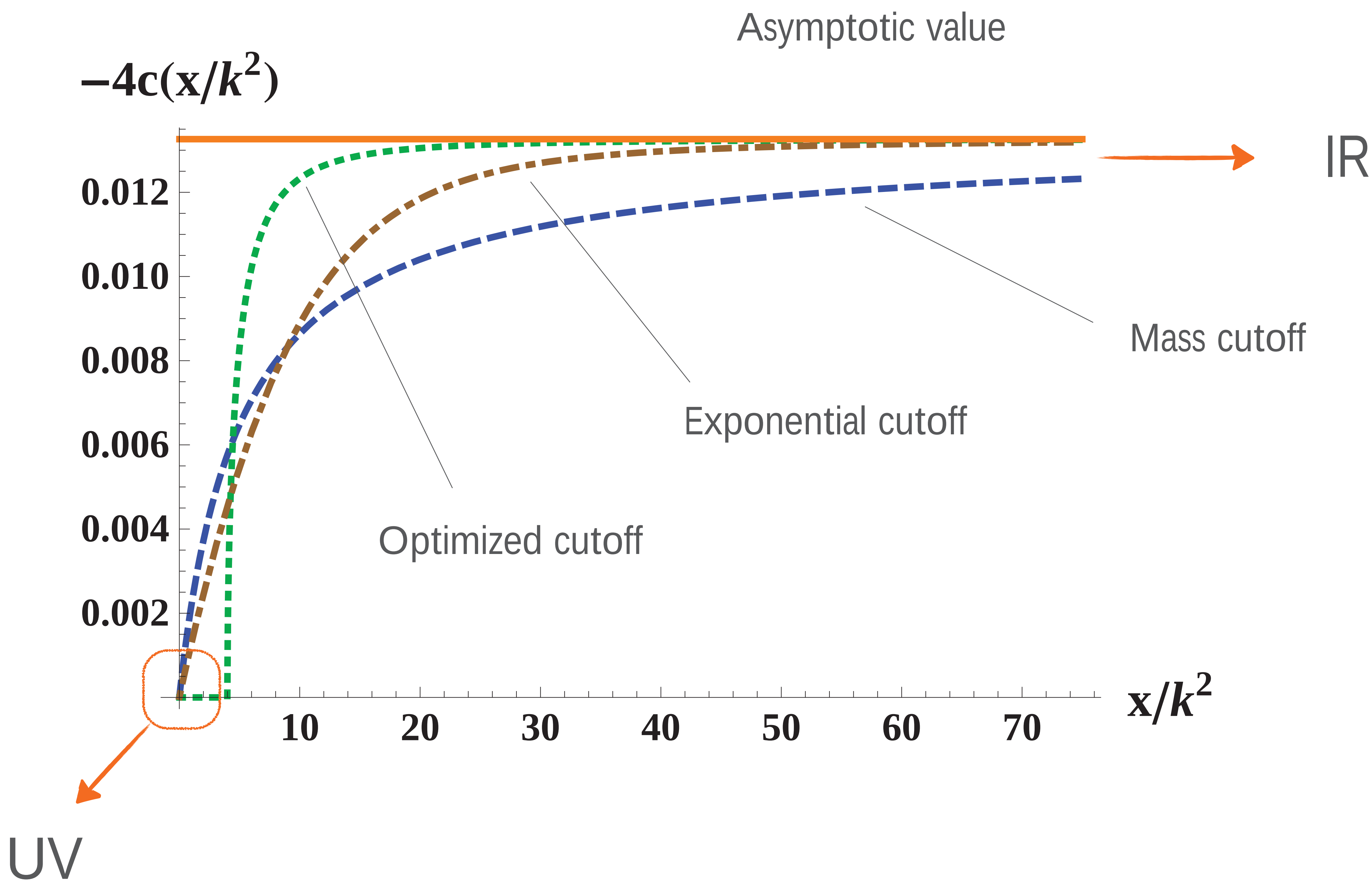
conformal variation of the EAA gives the <u>k-dependent trace anomaly</u>:

$$\langle T^\mu_\mu \rangle^I_k = -\frac{2}{\sqrt{g}} g_{\mu\nu} \frac{\delta \Gamma^I_u}{\delta g_{\mu\nu}} = -4c(\tilde{\Delta})R - \frac{2}{k^2} \sum_{n=0}^{\infty} \sum_{k=1}^{n-1} c_n \left(\frac{1}{\tilde{\Delta}^k} R \right) \left(\frac{1}{\tilde{\Delta}^{n-k}} R \right)$$

$$c_k(\Delta) = \frac{1}{\Delta} c(\tilde{\Delta}) \qquad \tilde{\Delta} = \Delta / k^2$$

and likewise for the Weyl-invariant EAA:

$$\langle T^\mu_\mu \rangle^{\mathrm{II}}_u = -\frac{2}{\sqrt{g}} g_{\mu\nu} \frac{\delta \Gamma^{II}_u}{\delta g_{\mu\nu}} = -4c\left(\frac{\mathcal{O}}{u^2}\right)\mathcal{R} - \frac{2}{u^2 \chi^2} \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} c_n \left(\frac{u^{2k}}{\mathcal{O}^k} \mathcal{R} \right) \left(\frac{u^{2(n-k)}}{\mathcal{O}^{n-k}} \mathcal{R} \right)$$


Asymptotic value
−4c(x/k²)
IR
0.012
0.010
0.008
0.006
0.004
0.002
Mass cutoff
Exponential cutoff
Optimized cutoff
x/k²
10
20
30
40
50
60
70
UV

The c-anomaly in d=4

- Curvature squared expansion:

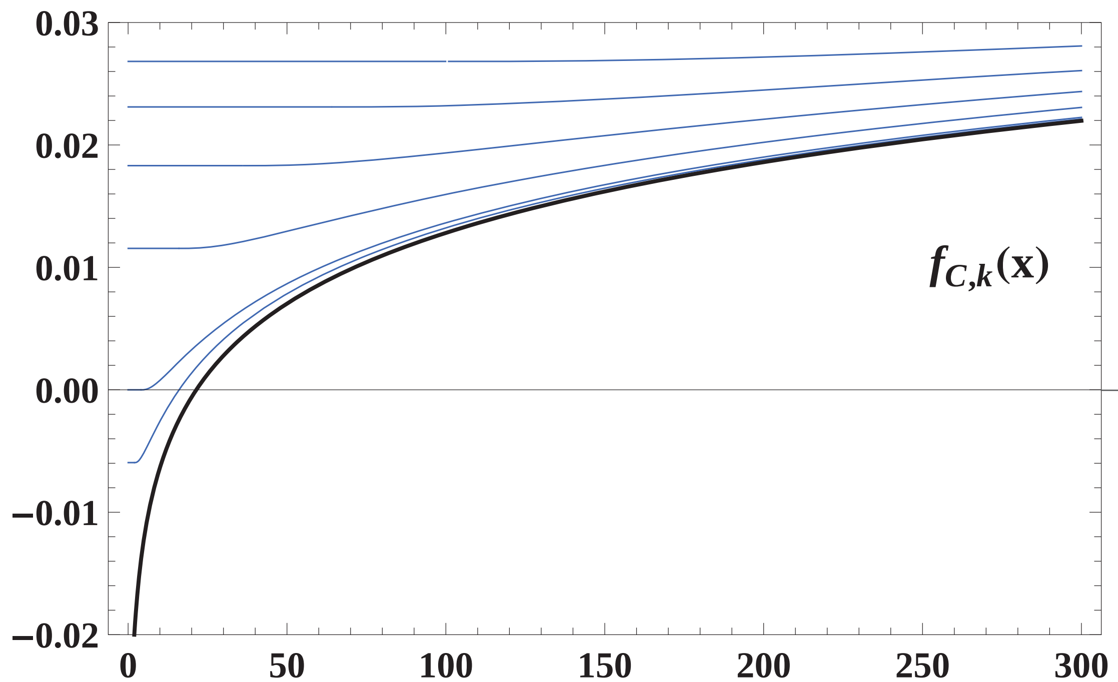
$$\Gamma^I_k = \int d^4x \sqrt{g}\, [a_k + b_k R + R f_{R,k}(\Delta) R + C_{\mu\nu\rho\sigma} f_{C,k}(\Delta) C^{\mu\nu\rho\sigma} + O(R^3)]$$

goes to 0

goes to one-loop EA

$$\Gamma^I = -\frac{1}{2}\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\, \frac{n_S + 6n_D + 12n_M}{120}\, C_{\mu\nu\rho\sigma} \log\left(\frac{\Delta}{k_0^2}\right) C^{\mu\nu\rho\sigma}$$

Deser-Schwimmer action



shape of the form-factor towards the IR

- Another action that generates the full anomaly is the Riegert action:

$$W(g_{\mu\nu}) = \int d^4x \sqrt{g}\frac{1}{8}\left(E - \frac{2}{3}\Box R\right)\Delta_4^{-1}\left[2c\,C^2 - a\left(E - \frac{2}{3}\Box R\right)\right] + \frac{a}{18}R^2$$

- BUT: wrong flat spacetime limit of TT-correlator.

- By using the WZ action in the relation previously found

$$\Gamma^I(g_{\mu\nu}) = \Gamma^{II}(g_{\mu\nu},\chi) - \Gamma_{WZ}(g_{\mu\nu},\chi)$$

we reobtain the Riegert action, e.g., from the equation of motion of the dilaton. Other choices give Riegert + Weyl-invariant terms

- One can compute the Weyl-invariant EAA for the c-term. In the IR limit this gives

$$\Gamma^{II}(g_{\mu\nu},\chi) = -\frac{1}{2}\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\,\frac{n_S + 6n_D + 12n_M}{120}\,\mathcal{C}_{\mu\nu\rho\sigma}\log\left(\frac{\mathcal{O}}{u_0^2}\right)\mathcal{C}^{\mu\nu\rho\sigma}$$

Interacting Matter

- For <u>free matter</u>, the trace anomaly is *zero* in flat space
- For <u>interacting matter</u>, with Weyl-invariant interactions of the form

$$S_{int}(g_{\mu\nu}, \Psi_a) = \lambda \int d^4x \sqrt{g}\, \mathcal{L}_{int}$$

dimensionless

one has in flat space:

$$\beta_\lambda = k\frac{d\lambda}{dk}$$

$$\int d^4x\, \omega \langle T^\mu{}_\mu \rangle = -\delta_\omega S_{int} = \int d^4x\, \omega\, \beta_\lambda \mathcal{L}_{int}$$

- <u>Two points of view</u>:
- **I**. Weyl invariance = zero beta functions.
- **II**. Change RG prescription: $\lambda \to \lambda(u) \qquad u = k/\chi$
- Then $\delta_\omega S_{int} = 0$ even if $\beta_\lambda \neq 0$

- The two terms in the fRG flow equation have the transformation properties:

$$\frac{\delta^2(\Gamma_k + \Delta S_k)}{\delta\phi\delta\phi} \mapsto \Omega^{-d-w} \frac{\delta^2(\Gamma_k + \Delta S_k)}{\delta\phi\delta\phi} \Omega^{-w} \ ,$$

$$k\frac{d}{dk}\frac{\delta^2 \Delta S_k}{\delta\phi\delta\phi} \mapsto \Omega^{-d-w} k\frac{d}{dk}\frac{\delta^2 \Delta S_k}{\delta\phi\delta\phi} \Omega^{-w}$$

field of weight w

- *Since the beta functional is Weyl invariant, if we start from an initial condition which is Weyl invariant, the EA will be Weyl invariant as well.*

Dynamical Gravity

- Usual Background Field Method, with background-split for metric and dilaton. We chose a constant dilaton background.

- Covariant measure for gravitons and gravity ghosts introduced exactly as before.

- Einstein-Hilbert truncation:

$$S(g,\chi) = \int d^4x\sqrt{g}\left[\lambda Z^2\chi^4 - \frac{1}{12}Z\chi^2\mathcal{R}\right]$$

$$\mathcal{R} = R - 6\chi^{-1}\Box\chi$$
$$Z\chi^2 = \frac{12}{16\pi G}$$
$$\lambda = \frac{2\pi}{9}G\Lambda$$

- Beta-functions agree with the known ones:

$$u\frac{dZ}{du} = \frac{1}{4\pi^2}\left(23 + 2n_M - n_D\right)u^2$$
$$u\frac{d\lambda}{du} = \frac{2+n_S+2n_M-4n_D}{32\pi^2 Z^2}u^2\left[u^2 - 16\lambda Z\frac{23+2n_M-n_D}{2+n_S+2n_M-4n_D}\right]$$

$$\tilde{\Lambda} = \Lambda/k^2 = 6\lambda Z/u^2$$
$$\tilde{G} = Gk^2 = 3u^2/4\pi Z$$

- General solution of this system:

boundary conditions

$$Z(u) = Z_0 + \frac{23 + 2n_M - n_D}{8\pi^2} u^2 \ ,$$

$$\lambda(u) = \frac{\pi^2((2 + n_S + 2n_M - 4n_D)u^4 + 128\pi^2 Z_0^2 \lambda_0)}{2(8\pi^2 Z_0 + (23 + 2n_M - n_D)u^2)^2}$$

- Gravitational fixed point corresponds to large-u behaviour (note that Z is *redundant*):

$$\lambda(u) \to \lambda_\star = \frac{\pi^2(2 + n_S + 2n_M - 4n_D)}{2(23 + 2n_M - n_D)^2}$$

$$Z(u) \to Z_* u^2 = \frac{23 + 2n_M - n_D}{8\pi^2} u^2$$

- At a FP, the dilaton *decouples*.

- The RG trajectory corresponding to the FP is the one with the boundary condition for Z set to zero. In this case, we find a vanishing EA.

Conclusions

- With a dilaton one can construct a Weyl invariant measure and EA
- This can be generalized to interacting matter using the EAA, and to dynamical gravity
- This construction clarifies the meaning of Weyl invariance in an RG context
- It helps clarifying some ambiguities for the trace anomaly in d=4

THANK YOU