Asymptotic Safety in the $f(R)$ approximation

Jürgen Dietz

University of Southampton

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Outline

1. The exact renormalisation group equation and $f(R)$ truncations for gravity

2. Fixed point equations and their solutions

3. Perturbations around fixed point solutions

4. Conclusions and Outlook

Exact renormalisation group equation (ERGE)

The path integral can be re-written as a functional differential equation:

$$\partial_t \Gamma_k[\varphi] = \frac{1}{2}\mathrm{Tr}\left[\left(\frac{\delta^2 \Gamma_k}{\delta\varphi\delta\varphi} + \mathcal{R}_k\right)^{-1} \partial_t \mathcal{R}_k\right]$$

Here $t = \ln \frac{k}{k_0}$ and

$$S_k[\varphi] = \frac{1}{2}\int d^4x\, \varphi \mathcal{R}_k \varphi$$

provides a cutoff mass for IR modes.

Cutoff functions

Via $S_k[\varphi]$ the cutoff operator $\mathcal{R}_k$ provides an effective k-dependent mass term. It has the structure

$$\mathcal{R}_k = \mathcal{Z}_k k^2 r\left(-\frac{D^2}{k^2}\right)$$

with $\lim_{x\to 0} r(x) = 1$ and $\lim_{x\to\infty} r(x) = 0$.

- introduces scheme dependence
- crucial for evaluating the ERGE
- optimised cutoff (Litim; '01): $r(x) = (1-x)\theta(1-x)$

Truncating theory space...

Evaluating the ERGE in general is as difficult as performing the functional integral.

$\Rightarrow$ Truncate theory space by making an ansatz for Γ_k

$$\Gamma_k = \int d^4x \sqrt{g} f_k(R)$$

where R is the scalar curvature of the metric $g_{\mu\nu}$.

Previous evidence for Asymptotic Safety was found with truncations like

$$f_k(R) = \sum_{n=0}^{N} u_n(k) R^n \quad \text{and} \quad f_k(R) = \sum_{n=0}^{2} u_n(k) R^n + \sigma(k) C^2 + \theta(k) R_{\mu\nu} R^{\mu\nu}.$$

$f(R)$-truncations

Three of this kind of FP equations have been derived by working on 4-spheres:

- Machado and Saueressig; 2008
- Codello, Percacci, Rahmede; 2009
- Benedetti and Caravelli; 2012

⇒ What does the space of solutions look like?

⇒ If there are fixed-point (FP) solutions, what are the eigen operators?

FP equation by Machado and Saueressig (2008)

$$
\begin{aligned}
&768\pi^2\left(2f - Rf'\right) = \\
&\quad \left[5R^2\theta\left(1-\tfrac{R}{3}\right) - \left(12 + 4\,R - \tfrac{61}{90}\,R^2\right)\right]\left[1-\tfrac{R}{3}\right]^{-1} + \Sigma \\
&+\left[10\,R^2\,\theta(1-\tfrac{R}{4}) - R^2\,\theta(1+\tfrac{R}{4}) - \left(36 + 6\,R - \tfrac{67}{60}\,R^2\right)\right]\left[1-\tfrac{R}{4}\right]^{-1} \\
&+\left[(2f' - 2Rf'')\left(10 - 5R - \tfrac{271}{36}R^2 + \tfrac{7249}{4536}R^3\right) + f'\left(60 - 20R - \tfrac{271}{18}R^2\right)\right]\left[f + f'(1-\tfrac{R}{3})\right]^{-1} \\
&+\tfrac{5R^2}{2}\left[(2f' - 2Rf'')\left\{r(-\tfrac{R}{3}) + 2r(-\tfrac{R}{6})\right\} + 2f'\theta(1+\tfrac{R}{3}) + 4f'\theta(1+\tfrac{R}{6})\right]\left[f + f'(1-\tfrac{R}{3})\right]^{-1} \\
&+\Big[(2f' - 2Rf'')f'\left(6 + 3R + \tfrac{29}{60}R^2 + \tfrac{37}{1512}\,R^3\right) \\
&\qquad -2Rf'''\left(27 - \tfrac{91}{20}R^2 - \tfrac{29}{30}\,R^3 - \tfrac{181}{3360}\,R^4\right) \\
&\qquad +f''\left(216 - \tfrac{91}{5}R^2 - \tfrac{29}{15}R^3\right) + f'\left(36 + 12R + \tfrac{29}{30}R^2\right)\Big]\left[2f + 3f'(1-\tfrac{2}{3}R) + 9f''(1-\tfrac{R}{3})^2\right]^{-1}
\end{aligned}
$$

where $\Sigma = 10\,R^2\,\theta\left(1-\frac{R}{3}\right)$.

Are there any solutions?

- This is a third order ordinary differential equation (ODE) $\Rightarrow$ three dimensional parameter space of solutions
- Most of parameter space will be ruled out by **moveable singularities** originating from non-linearity of ODE
- Each **fixed singularity** reduces the number of free parameters

Fixed singularities and parameter space

Suppose we have a normal form

$$f'''(R) = \frac{F(f, f', f'', R)}{R}$$

with a fixed singularity at $R = 0$.

Substituting in the Taylor expansion
$f(R) = a_0 + a_1 R + a_2 R^2 + \dots$ we find a Laurent series

$$\text{regular in } R = \frac{u(a_0, a_1, a_2)}{R} + \text{regular in } R,$$

with $u(a_0, a_1, a_2)$ being a non-trivial constraint on the three parameters a_0, a_1, a_2.

Origin of fixed singularities in FP equation

$$768\pi^2\left(2f - Rf'\right) =$$

$$\left[5R^2\theta\left(1-\tfrac{R}{3}\right) - \left(12 + 4\,R - \tfrac{61}{90}\,R^2\right)\right]\left[\mathbf{1} - \tfrac{\mathbf{R}}{\mathbf{3}}\right]^{-\mathbf{1}} + \Sigma$$

$$+\left[10\,R^2\,\theta(1-\tfrac{R}{4}) - R^2\,\theta(1+\tfrac{R}{4}) - \left(36 + 6\,R - \tfrac{67}{60}\,R^2\right)\right]\left[\mathbf{1} - \tfrac{\mathbf{R}}{\mathbf{4}}\right]^{-\mathbf{1}}$$

$$+\left[(2f' - 2Rf'')\left(10 - 5R - \tfrac{271}{36}R^2 + \tfrac{7249}{4536}R^3\right) + f'\left(60 - 20R - \tfrac{271}{18}R^2\right)\right]\left[f + f'(1-\tfrac{R}{3})\right]^{-1}$$

$$+\tfrac{5R^2}{2}\left[(2f' - 2Rf'')\left\{r(-\tfrac{R}{3}) + 2r(-\tfrac{R}{6})\right\} + 2f'\theta(1+\tfrac{R}{3}) + 4f'\theta(1+\tfrac{R}{6})\right]\left[f + f'(1-\tfrac{R}{3})\right]^{-1}$$

$$+\left[(2f' - 2Rf'')f'\left(6 + 3R + \tfrac{29}{60}R^2 + \tfrac{37}{1512}\,R^3\right)\right.$$

$$-2f'''\mathbf{R}\left(\mathbf{27} - \tfrac{\mathbf{91}}{\mathbf{20}}\mathbf{R}^{\mathbf{2}} - \tfrac{\mathbf{29}}{\mathbf{30}}\,\mathbf{R}^{\mathbf{3}} - \tfrac{\mathbf{181}}{\mathbf{3360}}\,\mathbf{R}^{\mathbf{4}}\right)$$

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- We find single poles at $\mathbf{R_c = 0, 2.0065, 3, 4}$.
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Second equation (Codello, Percacci, Rahmede; '09) has all θ-functions set to one and a different Σ

$\Rightarrow$ the same single poles $\Rightarrow$ same conclusion holds

Fixed point equation by Benedetti & Caravelli (2012)

They found a FP equation with the following structure:

$$f'''(R) = \frac{\mathcal{N}(f, f', f'', R)}{R\,(R^4 - 54R^2 - 54)\,((R-2)f'(R) - 2f(R))}$$

This has fixed singularities $R_c = 0$ and $R_c = R_+ \approx 7.414$.

New cutoff implementation eliminates previous fixed singularities but creates $R_c = R_+$.

Not taking possible asymptotic ($R \to \infty$) constraints into account we would expect **one-dimensional sets of solutions** in the range $R \geq 0$.

Asymptotic Expansion

Leading asymptotic behaviour can be found by substituting the ansatz

$$f(R) = AR^p + O(R^{p-1})$$

into the fixed point equation.

$\Rightarrow$ for asymptotic equality we find $p = 2$

- Uncovering other contributions: $f(R) = AR^2 + \delta f(R)$
- Leads to: $R^3 \delta f''' - R^2 \delta f'' + 6R\delta f' - 10\delta f = 0$
- **Additional solutions:** $\delta B\, R \cos \ln R^2 + \delta C\, R \sin \ln R^2$

$\Rightarrow$ We have **three** parameters A, B, C in the asymptotic expansion

Systematics of the asymptotic expansion

Introducing a book-keeping parameter ϵ through

$$f_\epsilon(R) = \epsilon^2 f(R/\epsilon),$$

the asymptotic series assumes the form

$$f_\epsilon(R) = R^2 g_0(R) + \epsilon R g_1(R) + \epsilon^2 g_2(R) + \dots$$

and we consider the limit $\epsilon \to 0$.

- We know already: $g_0(R) = A$
- Use $f_\epsilon(R)$ and solve order by order in $\epsilon \Rightarrow g_n(R)$
- Thus we get, e.g. $g_1(R) = \frac{3}{2}A + B \cos \ln R^2 + C \sin \ln R^2$

Higher orders in the asymptotic expansion

- We can solve analytically for $g_2(R)$ but not anymore for higher orders $g_3(R), \ldots$
- Denominators appear containing a factor depending on A, B, C
- These singularities can be avoided if

$$\frac{121}{20} A^2 > B^2 + C^2.$$

- This cone condition is the **only** restriction provided by the asymptotics

Asymptotic expansion - summary

As $R \to \infty$ the fixed point function behaves as

$$f(R) = A\,R^2 + R\left\{\frac{3}{2}A + B\cos\ln R^2 + C\sin\ln R^2\right\} + O(R^0).$$

Lower orders depend on $\cos\ln R^2$, $\sin\ln R^2$ and all three constants A, B, C.

To guarantee the solution is singularity free for large R we need

$$\frac{121}{20}A^2 > B^2 + C^2.$$

According to parameter counting, we still expect **lines** of fixed point solutions.

Bridging singularities

We match across the fixed singularities $R = 0, R_+$ using Taylor expansions of form

$$f(R) = b_0 + b_1(R - R_+) + \sum_{n=2}^{5} \frac{b_n(b_0, b_1)}{n!}(R - R_+)^n,$$

$$f(R) = a_0 + a_1 R + \sum_{n=2}^{5} \frac{a_n(a_0, a_1)}{n!} R^n.$$

Fixed-point equation determines higher order coefficients $n = 2, 3, \ldots$.

Strategy for finding solutions

- Start with an initial pair (a_0, a_1) and calculate $f(\epsilon), f'(\epsilon), f''(\epsilon)$ using the Taylor series at $R = 0$

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- If second derivatives match we have found a solution for $0 \leq R \leq R_+$

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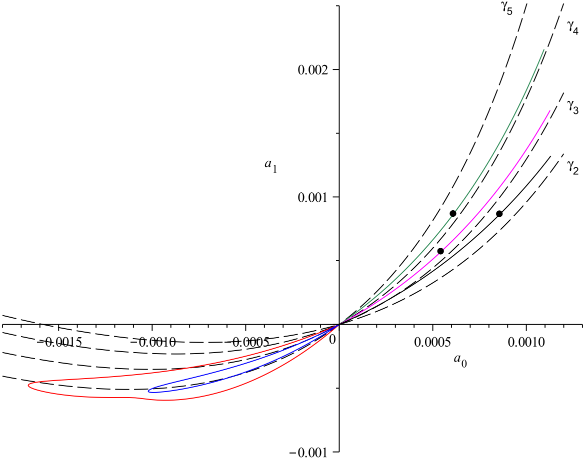
- If second derivatives match we have found a solution for $0 \leq R \leq R_+$
- For any such solution integrate from $R_+ + \epsilon$ up to some R_∞ and match to asymptotic expansion
- If A, B, C are inside the cone we have found a solution for all $R \geq 0$.

Solution lines in the a-plane



Linearising around the FP

Write

$$f(R,t) = f(R) + \delta f(R,t)$$

with

$$\delta f(R,t) = \alpha v(R) \exp(-2\lambda t).$$

The full ERGE then provides a **linear** third order ODE for $v(R)$.

- The fixed singular points $R = 0, R_+$ provide two constraints and a third is given by normalising $v(R)$
- Large R expansion for $v(R)$ does not provide any constraint on the parameters

$\Rightarrow$ **for each λ we get a discrete set of solutions for $v(R)$.**

What about $R \leq 0$?

- So far we have considered these equations only for $\mathbf{R \geq 0}$
- However: Quantum Gravity must make sense on negatively curved spaces as well!
- $\Rightarrow$ We can include the range $R \leq 0$ by analytic continuation
- This gives us another fixed singularity at $R_- = -R_+$
- $\Rightarrow$ **Discrete** set of fixed points, each with **quantised** eigenspectrum

Global solutions valid for all R?

- None of the solutions for negative R extends to $R = -\infty$
- We found three fixed point solutions valid on $R_{-} \leq R < \infty$

What needs to be done:
Derivation of flow equation on spaces of arbitrary scalar curvature R, including $R \leq 0$.

Conclusions

- Parameter counting provides a powerful tool for these equations
- Full asymptotic expansion contains three parameters which are constrained to lie inside a cone
- FP equations can be solved by a combination of analytical and numerical methods
- The range $R \geq 0$ in the $f(R)$ approximation is not enough to make sense out of Asymptotic Safety
- Analytic continuation to $R < 0$ of fixed point equation derived for $R > 0$ is unsatisfactory

Example fixed-point functions