A bootstrap towards asymptotic safety

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Introduction

Quantum fluctuations cause the coupling constants of every field theory to depend on the energy at which they are probed.

It is believed that in order a field theory to exist fundamentally, its couplings should approach a fixed point at the UV.

Two different cases distinguished by the value of the couplings at the fixed point.

Asymptotic freedom

- Gaussian fixed point $g_* = 0$
- Non-interacting theory
- ex. Q.C.D.

Asymptotic safety

- Non-gaussian fixed point $g_* \neq 0$
- Interacting theory
- ex. Quantum gravity?

Asymptotic freedom

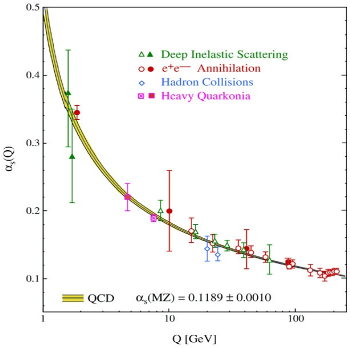
The standard paradigm for an asymptotically free theory is Q.C.D. with a fundamental action given by

$$S = \int d^4x \; \bar{\psi}(i\gamma_\mu D^\mu - m)\psi - \frac{1}{2g_s^2}\mathrm{tr}\{\mathrm{F}_{\mu\nu}\mathrm{F}^{\mu\nu}\}$$

- Beta function for field strength $\alpha_s \equiv \frac{g_s^2}{4\pi}$

$$\beta(\alpha_s) = -\left(11 - \frac{n_f}{3}\right)\frac{\alpha_s}{2\pi}$$

- $\alpha_s = 0$ is a gaussian fixed point.
- Because the theory is non-interacting we know *exactly* which are the relevant, marginal and irrelevant operators in the UV.
- Canonical power counting is applicable.



Asymptotic safety

- The UV flow of the theory is governed by an *interacting* fixed point

$$g_* \neq 0$$

- There are trajectories connecting the UV regime with the classical regime.
- An interacting fixed point implies that the couplings can acquire anomalous dimensions.
- The scaling of invariants at the fixed point can differ significantly from the canonical scaling.
- We do not know a-priori which operators are relevant, marginal or irrelevant at the UV.

Challenges

Consider that we have an effective theory which is consistent with a large set of operators

$$\{\mathcal{O}_i\}, \qquad i = 1 \ldots n$$

- If we have to rely on approximations to solve the system which operators should we keep?
- What is the set of UV relevant and UV irrelevant operators?
- Is there any organisation principle analogous to the canonical power counting for asymptotically free theories?

We adopt a bootstrap approach to the problem by successively adding new operators and iteratively checking the UV properties of the theory. Thus we associate the effects on fixed points and scaling exponents with each new operator added and consistently check how the validity of the approximation is affected.

Weinberg's conjecture

$$S = \int d^4x \sqrt{g}\, \lambda_i\, \mathcal{O}_i$$

with $[\lambda_i] = d_i$, $[\mathcal{O}_i] = 4 - d_i$ and dimensionless couplings defined as $\bar{\lambda}_i = k^{-d_i}\lambda_i$.

Fixed point condition:

$$\boxed{\beta_i(\bar{\lambda}^*) = 0}$$

- Linearise the flow around the fixed point:

$$\beta_i(\bar{\lambda}) = \sum_j \mathbb{B}_{ij}(\bar{\lambda}_j - \bar{\lambda}_j^*) + O([\bar{\lambda} - \bar{\lambda}^*]^2)$$

  with $\mathbb{B}_{ij} = \partial\beta_i(\bar{\lambda})/\partial\bar{\lambda}_j\big|_{\bar{\lambda}=\bar{\lambda}^*}$
- The general solution reads:

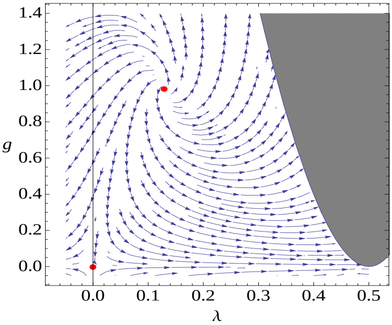
$$\bar{\lambda}_i(k) = \bar{\lambda}^* + \sum_S c_S V_i^S k^{\theta_S}$$

- The beta functions have the general form:

$$\beta_i = -d_i\, \bar{\lambda}_i + \text{quantum corrections}$$

For $\theta_S > 0$ the theory flows to the FP in the UV only if the corresponding c_S vanish. This requirement defines the *UV critical surface*.

- Non-perturbative results in $2+\epsilon$ dimensions (Kawai, Ninomiya '90)
- First fixed point in 4 dimensions (Souma '99)



$g = G_N\, k^2 \qquad ; \qquad \lambda = \Lambda/k^2$

UV fixed point at $\boxed{g = 0.984,\ \lambda = 0.129}$

Gaussian fixed point $\boxed{g = 0,\ \lambda = 0}$

- $\Gamma_k = \int d^4x \frac{1}{16\pi G_k}(2\Lambda_k - R)$
  Reuter ('96)
- $\Gamma_k = \int d^4x \frac{1}{16\pi G_k}(2\Lambda_k - R + g_2 R^2)$
  Lauscher and Reuter ('02)
- $\Gamma_k = \int d^dx \frac{1}{16\pi G_k}(2\Lambda_k - R)$
  Fischer and Litim ('06)
- $\Gamma_k = \int d^4x(\lambda_0 + \lambda_1 R + \cdots + \lambda_8 R^8)$
  Codello, Percacci, Rahmede ('09)
- $\Gamma_k = \int d^4x(\lambda_0 + \lambda_1 R + \lambda_2 R^2 + \lambda_3 C^2)$
  Benedetti, Machado, Saueressig ('09)

Open questions

- How higher order operators affect the fixed point structure?
- Does the picture stabilises or it breaks down?
- What is the radius of convergence for the expansion?
- Do we always get finite number of negative eigenvalues?

The bootstrap

- Choose as an ansatz a polynomial expansion in the Ricci scalar.
- Successively add higher order operators.
- Consistently check for the UV fixed point.
- Consistently check that we get the same number of negative eigenvalues.
- Organise the eigenvalues according to their values.
- Check if the polynomial expansion is indeed a good approximation.

The calculation setup

- We use the Wetterich equation for the effective action

$$\partial_t \Gamma_k[\varphi] = \frac{1}{2}\mathrm{Tr}\left[\partial_t R_k \cdot \left(\Gamma_k^{(2)}[\varphi] + R_k\right)^{-1}\right]$$

- We choose our gravitational anstatz to be an arbitrary function of the Ricci scalar

$$\Gamma_k^{\mathrm{grav}} = \int d^4x\,\sqrt{g}\,F_k(R)$$

- Choose the regulator profile function

$$\mathcal{R}_k(y) = (k^2 - y)\theta(k^2 - y).$$

- Decompose the fluctuations into the TT decomposition:

$$h_{\mu\nu} = h_{\mu\nu}^T + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu + \nabla_\mu \nabla_\nu \sigma - \frac{1}{4} g_{\mu\nu} \nabla^2 \sigma + \frac{1}{4} g_{\mu\nu} h \ .$$

- Compute the trace using heat kernel techniques, choosing our background to be a sphere.
- Define dimensionless variables as:

$$\rho = \frac{R}{k^2} \qquad ; \qquad f_k(\rho) = \frac{F_k(R)}{k^4}$$

The flow equation

$$384\pi^2 \underbrace{\left(4f(\rho) - 2\rho f'(\rho) + \partial_t f(\rho)\right)}_{\partial_t \Gamma_k} = \underbrace{\frac{A}{2f(\rho) + (3-2\rho)f'(\rho) + (\rho-3)^2 f''(\rho)}}_{hh}$$

$$+ \underbrace{\frac{B}{3f(\rho) - (\rho-3)f'(\rho)}}_{h^T h^T} + \underbrace{C}_{\texttt{all the rest}}$$

with

$$A = \left(48 + 18\rho + \frac{29}{15}\rho^2 + \frac{37}{756}\rho^3\right) f'(\rho) + \left(216 - 12\rho - \frac{121}{5}\rho^2 - \frac{29}{10}\rho^3 - \frac{37}{756}\rho^4\right) f''(\rho)$$
$$+ \left(-54\rho + \frac{91}{10}\rho^3 + \frac{29}{15}\rho^4 + \frac{181}{1680}\rho^5\right) f'''(\rho) + \left(6 + 3\rho + \frac{29}{60}\rho^2 + \frac{37}{1512}\rho^3\right) \partial_t f'(\rho)$$
$$+ \left(27 - \frac{91}{20}\rho^2 - \frac{29}{30}\rho^3 - \frac{181}{3360}\rho^4\right) \partial_t f''(\rho)$$

$$B = \left(240 - 90\rho - \frac{1}{3}\rho^2 + \frac{311}{756}\rho^3\right) f'(\rho) + \left(-60\rho + 30\rho^2 + \frac{1}{6}\rho^3 - \frac{311}{756}\rho^4\right) f''(\rho)$$
$$+ \left(30 - 15\rho - \frac{1}{12}\rho^2 + \frac{311}{1512}\rho^3\right) \partial_t f'(\rho)$$

$$C = \frac{-576 + 60\rho - \frac{4537}{15}\rho^2 + \frac{209}{2}\rho^3}{\rho^2 - 7\rho + 12}$$

The iterative solution

- In practice we solve this by making a polynomial expansion of $f(\rho)$ up to order N. The coefficients λ_n are the running couplings of the theory

$$f(\rho) = \sum_{n=0}^{n=N} \frac{1}{n!} f^{(n)} \rho^n = \sum_{n=0}^{n=N} \lambda_n \rho^n$$

- Observe that at each order the highest coefficient is $f^{(N+2)}$ and it always comes linearly. Thus, we can solve to get a relation of the form

$$\boxed{f^{(N+2)} = \mathcal{F}\left(f^{(0)}, f^{(1)}, \ldots, f^{(N+1)}\right)}$$

- This allows us to start from $n = 0$ and iteratively solve to get

$$\begin{aligned} f^{(N+1)} &= \mathcal{F}_{N+1}(f^{(0)}, f^{(1)}) \\ f^{(N+2)} &= \mathcal{F}_{N+2}(f^{(0)}, f^{(1)}) \end{aligned}$$

- Using as boundary conditions:

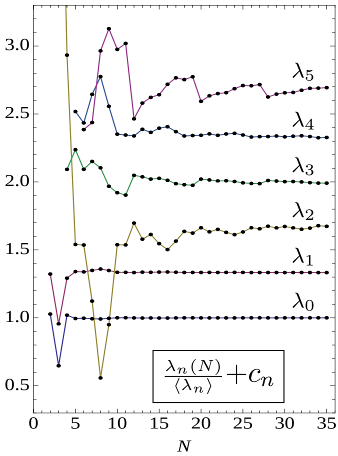
$$\begin{aligned} f^{(N+1)} &= 0 \\ f^{(N+2)} &= 0 \end{aligned}$$

  we find $f^{(0)}$ and $f^{(1)}$ and consequently all the other couplings.

- Using a computer algorithm we extend the computation to order

$$\boxed{N = 35}$$

Fixed point values



The FP values for the first 6 couplings.

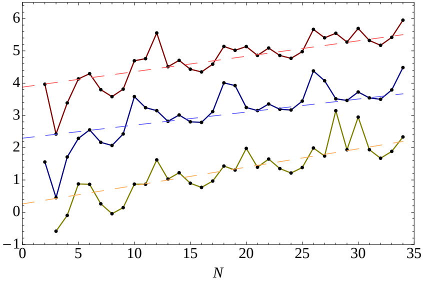
The standard deviations for the last 8 orders are

$$\langle\lambda_0\rangle = 0.0050845 \pm 0.0145\%$$
$$\langle\lambda_1\rangle = -0.020441 \pm 0.0266\%$$
$$\langle\lambda_2\rangle = 0.0003097 \pm 0.897\%$$
$$\langle\lambda_3\rangle = -0.008862 \pm 0.699\%$$
$$\langle\lambda_4\rangle = -0.007297 \pm 0.508\%$$
$$\langle\lambda_5\rangle = -0.004659 \pm 2.443\%$$

Observe the 8-fold periodicity for first time

Associated with singularities in the complex plane

Convergence of the FPs

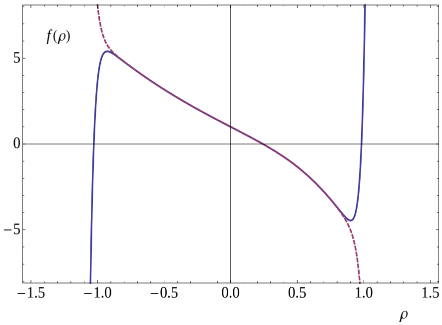


The convergence check $-\log\left|1-\frac{\lambda_n}{\lambda_{35}}\right|$ for the first 3 couplings

- $\lambda_0 \to$ blue line
  slope: ~ 0.04
  25 orders for one significant digit
- $\lambda_1 \to$ red line $-$ 2
  slope: ~ 0.048
  20 orders for one significant digit
- $\lambda_2 \to$ yellow line
  slope: ~ 0.057
  17 orders for one significant digit

The radius of convergence

The form of the function $f(\rho)$ for $N = 30$ (dashed line) and $N = 34$ (blue line).



The radius of convergence is estimated to be $\rho \sim 0.82$.

There are no de-Sitter solutions within the radius of convergence

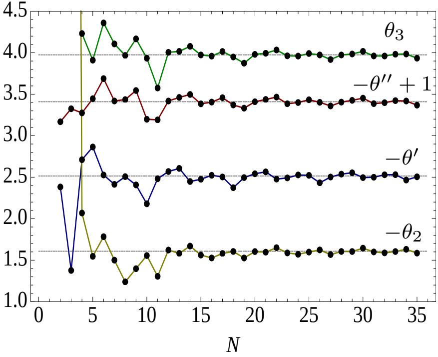
The eigenvalues

At each successive order of truncation we *always* find 3 negative eigenvalues

A pair of complex conjugates $\theta = \theta' \pm \theta''$ and a real one θ_2

Starting from θ_3 all the eigenvalues are positive.

The 8-fold periodicity is observed also in the convergence of eigenvalues



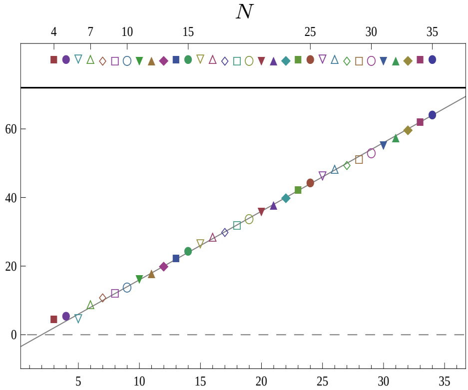
The first four eigenvalues.

$$\langle \theta' \rangle = -2.51 \pm 1.2\%$$
$$\langle \theta'' \rangle = -2.41 \pm 1.1\%$$
$$\langle \theta_2 \rangle = -1.61 \pm 1.3\%$$
$$\langle \theta_3 \rangle = 3.97 \pm 0.6\%$$

Revisiting Weinberg's argument

$$\beta_i = (2n - 4)\lambda_i + \text{Quantum corrections}$$

The highest eigenvalues at each order of the approximation



Near-gaussianity

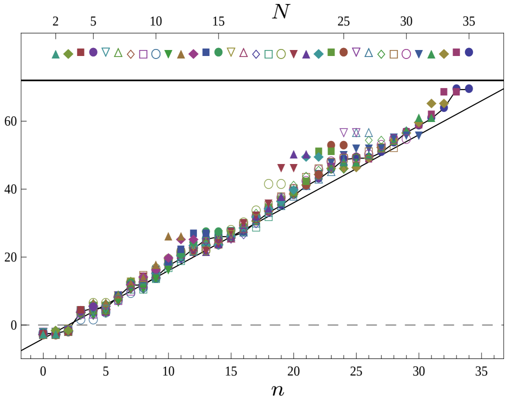
All the eigenvalues for every order in the approximation

We make a least-square fit for the last 24 data sets

$$\vartheta_n \approx \alpha \cdot n - b$$

with the non-perturbative coefficients at the UV fixed point

$$\alpha_{UV} = 2.17 \pm 5\%$$
$$b_{UV} = 4.06 \pm 10\%$$



Conclusions

- Extensive investigation for $f(R)$ quantum gravity up to order $N = 35$.
- Consistency for the fixed points and the critical exponents at every order of the approximation.
- Inclusion of more operators seems to stabilise the picture.
- The critical exponents tend to take near-gaussian values.

⋆ *Open challenges*

- Look for trajectories to connect with IR physics.
- Inclusion of more operators, C^2, $R_{\mu\nu}R^{\mu\nu}$ ....