Modulated Ground State of Gravity Theories with Stabilized Conformal Factor

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Main motivation

- What is the ground state of the theory?
- Flow equation is not useful from this point of view
- Is the vacuum stable/unstable or metastable?
- Is the Lauscher-Reuter-Wetterich stabilization mechanism still plausible here?

Conformal factor instability

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Instability of flat space at finite temperature

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The instabilities of quantum gravity are investigated using the path-integral formulation of Einstein's theory. A brief review is given of the classical gravitational instabilities, as well as the stability of flat space. The Euclidean path-integral representation of the partition function is employed to discuss the instability of flat space at finite temperature. Semiclassical, or saddle-point, approximations are utilized. We show how the Jeans instability arises as a tachyon in the graviton propagator when small perturbations about hot flat space are considered. The effect due to the Schwarzschild instanton is studied. The small fluctuations about this instanton are analyzed and a negative mode is discovered. This produces, in the semiclassical approximation, an imaginary part of the free energy. This is interpreted as being due to the metastability of hot flat space to nucleate black holes. These then evolve by evaporation or by accretion of thermal gravitons, leading to the instability of hot flat space. The nucleation rate of black holes is calculated as a function of temperature.

The conformal sector of $R + R^2$ gravity

Let us consider

$$\mathcal{S}[g_{\mu\nu}] = \int d^4x\sqrt{g}\left\{\frac{1}{16\pi G}\left(-R + 2\Lambda\right) + \beta R^2\right\}$$

the cosmological constant Λ and the dimensionless parameter β are assumed positive throughout. We shall see that $\beta > 0$ guarantees that $\mathcal{S}$ is bounded below. All stationary points satisfy:

$$\frac{1}{16\pi G}\left[G_{\mu\nu} + \Lambda g_{\mu\nu}\right] + \beta\left[-(G_{\mu\nu} + R_{\mu\nu})R + 2D_\mu D_\nu R - 2g_{\mu\nu}D^2 R\right] = 0$$

with the Einstein tensor $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$. Contracting EOM with $g^{\mu\nu}$ yields

$$R - 4\Lambda + (96\pi G\beta)\; D^2 R = 0$$

The conformal sector of $R+R^2$ gravity

Rather than trying to find the global minimum of $\mathcal{S}[g_{\mu\nu}]$ defined over the space of all metrics we shall analyze the action restricted to the conformally flat metrics

$$g_{\mu\nu} = \phi^2(x)\delta_{\mu\nu}.$$

Writing $S[\phi] = \mathcal{S}[g_{\mu\nu} = \phi^2\delta_{\mu\nu}]$ for the functional depending on the conformal factor ϕ we obtain:

$$S[\phi] = \int d^4x \left\{ \frac{3}{8\pi G}\,\phi\Box\phi + \frac{\Lambda}{8\pi G}\,\phi^4 + 36\beta\left(\frac{\Box\phi}{\phi}\right)^2 \right\}$$

where we used that $\sqrt{g} = \phi^4$ and $R = -6\phi^{-3}\Box\phi$ with $\Box = \delta^{\mu\nu}\partial_\mu\partial_\nu$ for metrics of the form $g_{\mu\nu} = \phi^2\delta_{\mu\nu}$. If $\beta = 0$ the restricted above functional has the appearance of a scalar ϕ^4-action with a “wrong sign" kinetic term.

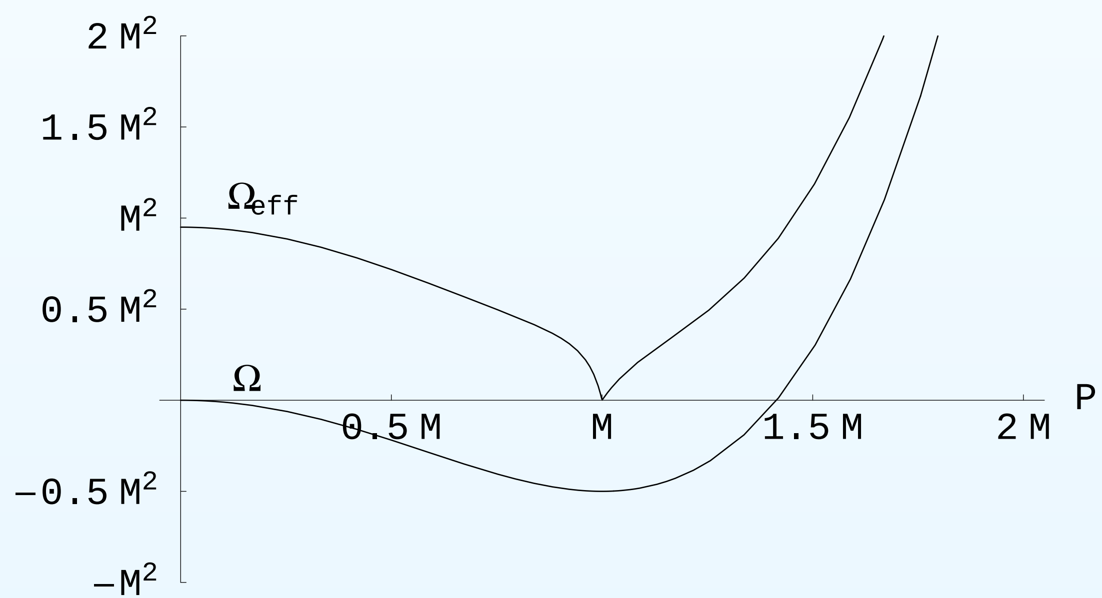
The main topic of the present work is the investigation of the global minimum action configuration(s) $\phi_{\min}$ of the restricted functional $S[\phi]$. It is plausible to assume that its essential qualitative features, in particular the existence of a modulated phase for certain parameter values, will be shared by the true minimum $g_{\mu\nu}^{\min}$, $i.e.$ that of $\mathcal{S}[g_{\mu\nu}]$ defined for all, not necessarily conformally flat metrics.

Lauscher Reuter and Wetterich model (PRD, 2000)

In this model the condensation of spatially inhomogeneous modes has been studied in detail within a scalar toy model which mimics certain features of $R + R^2$ gravity. It consists of a complex scalar field χ, on d-dimensional flat euclidean space, governed by the action

$$S[\chi] = \int d^d x \left\{ \chi^* \, \Omega(-\Box) \, \chi + \frac{\lambda}{2} \, |\chi|^4 \right\}$$

with $\Omega(-\Box) = \Box + \Box\Box/(2M^2)$ or, in momentum space, $\Omega(p^2) = -p^2 + \frac{(p^2)^2}{2M^2}$

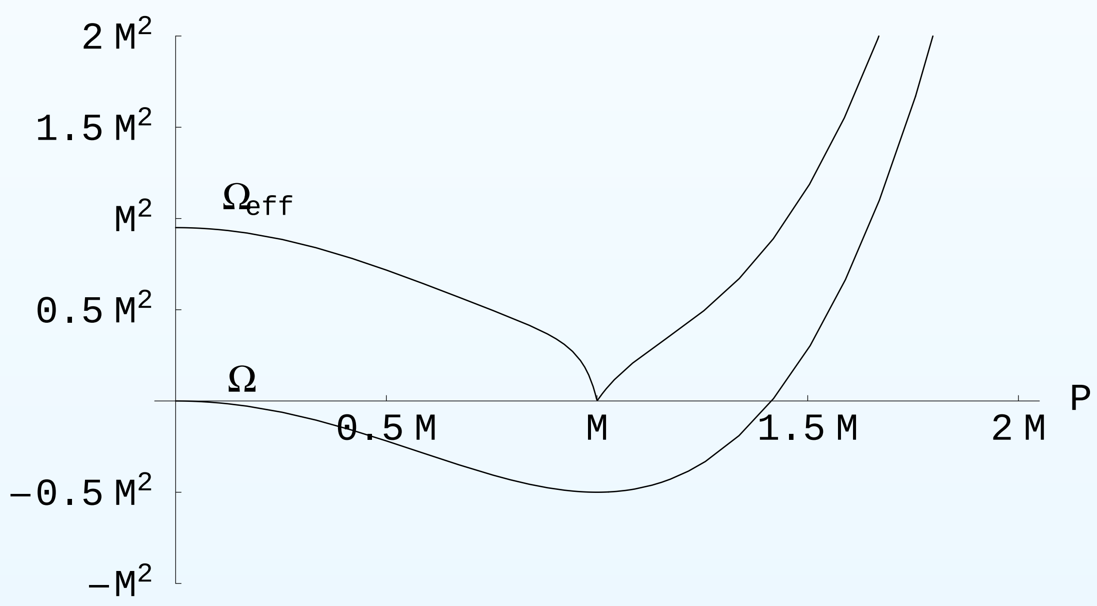


Reuter Lauscher and Wetterich model

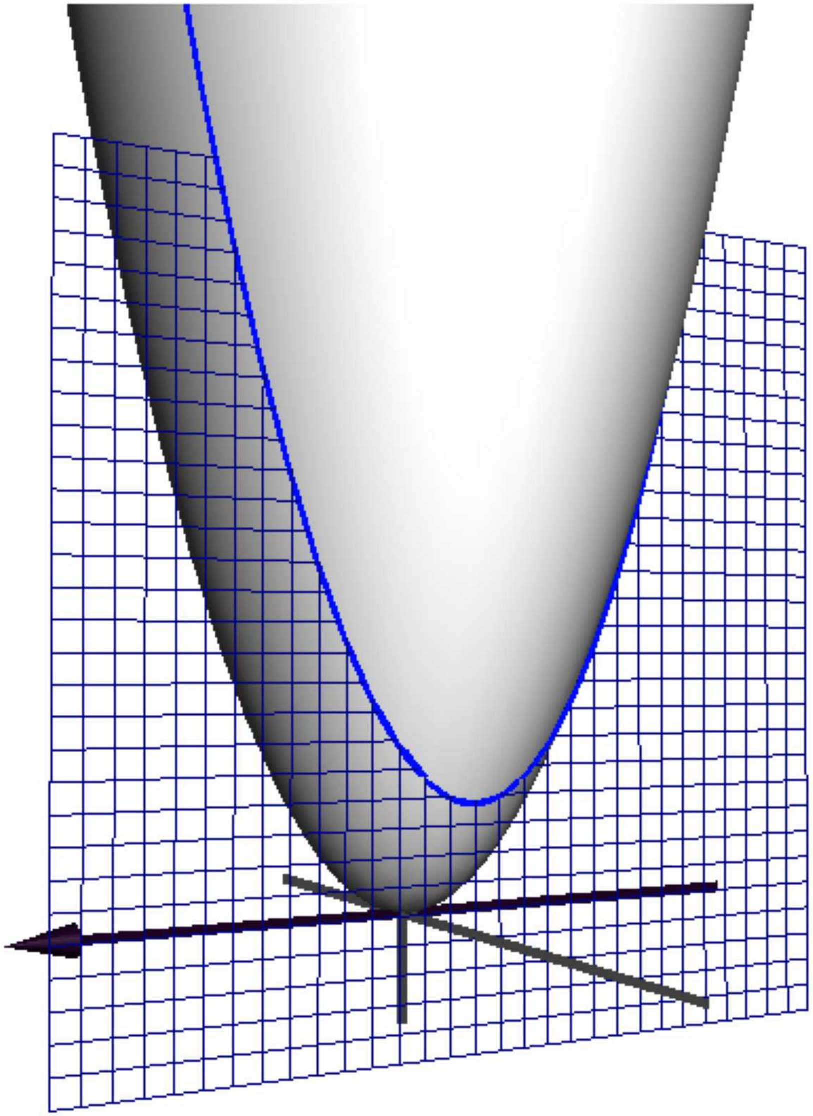
Exciting the χ-modes with momenta in the interval $p^2 \in [0, 2M^2]$ lowers the action. It was shown that the global minimum of the functional is given by the family of plane waves

$$\chi(x; n, \alpha) = \frac{M}{\sqrt{2\lambda}} \exp\left(iMn_\mu x^\mu + i\alpha\right)$$

labeled by a unit vector $n \in S^{d-1}$ and a phase angle α. The classical vacuum manifold is $S^{d-1} \times S^1$ therefore.



The conformal sector of $R + R^2$ gravity



The conformal sector of $R+R^2$ gravity

Sometimes it is advantageous to introduce the conformal factor in a slightly different way so that the second order kinetic term in S has a standard normalization (up to its sign). Writing

$$g_{\mu\nu}(x) = \frac{1}{3}(4\pi G)\,\chi^2(x)\,\delta_{\mu\nu}\,, \;\; i.e.\;\; \phi^2 \equiv \frac{1}{3}(4\pi G)\,\chi^2$$

the action for χ reads

$$S[\chi] = \int d^4x \Big\{\frac{1}{2}\,\chi\Box\chi + 36\beta\left(\frac{\Box\chi}{\chi}\right)^2 + \frac{u}{4!}\,\chi^4\Big\} \qquad (0.1)$$

$$= 36\beta\int d^4x\,\chi^4\left(\frac{\Box\chi}{\chi^3} + \frac{1}{144\beta}\right)^2 + \frac{\gamma}{576\beta}\int d^4x\,\chi^4 \qquad (0.2)$$

with the parameters

$$u \equiv \frac{16\pi}{3}G\Lambda, \quad \gamma = 128\pi\,\beta G\Lambda - 1 \equiv 24\,u\beta - 1$$

As $[G]=-2$, $[\Lambda]=+2$ the parameter u is dimensionless, the same is true for the new field variables: $[\chi]=0$. Since $\ell_{\rm Pl}\equiv\sqrt{G}\equiv m_{\rm Pl}^{-1}$ is the Planck length, we see that χ measures proper distances $ds^2=\frac{4\pi}{3}(\chi(x)\ell_{\rm Pl})^2\,\delta_{\mu\nu}dx^\mu dx^\nu$ in units of $\ell_{\rm Pl}$.

Special case $\gamma = 0$: The Bogomolny equation

Let us assume that $\gamma = 128\pi\beta G\Lambda - 1 = 0$ a Then $\mathcal{S}[g_{\mu\nu}] \geq 0$ and equality holds for metrics with

$$R(g) = 4\Lambda$$

The “Bogomolny equation": all solutions saturate the lower bound $\mathcal{S}[g_{\mu\nu}] = 0$. Conformally flat solutions to led to solve the Yamabe problem

$$\Box\chi + \frac{u}{6}\chi^3 = 0$$

The general plane wave type solution of involves two free constants, $\widehat{\chi}$ and α. It reads

$$\chi(x) = \widehat{\chi}\,\mathrm{sn}\Big(\widehat{\chi}\sqrt{\frac{u}{12}}\,n_\mu x^\mu + \alpha; i\Big)$$

Here sn denotes the $sinus\ amplitudinis$ with purely imaginary modulus $k = i$, and n^μ is an arbitrary unit vector, $\delta_{\mu\nu}n^\mu n^\nu = 1$. The solutions have a nontrivial periodicity in the direction of n_μ and are constant in the three directions perpendicular to it.

Special case $\gamma = 0$: The Bogomolny equation

If we employ the dimensionful coordinates

$$\bar{x}^\mu = \zeta\, \widehat{\chi} \sqrt{\frac{4\pi}{3}} \ell_{\rm Pl}\, x^\mu$$

the line element related to the minimum action configuration reads

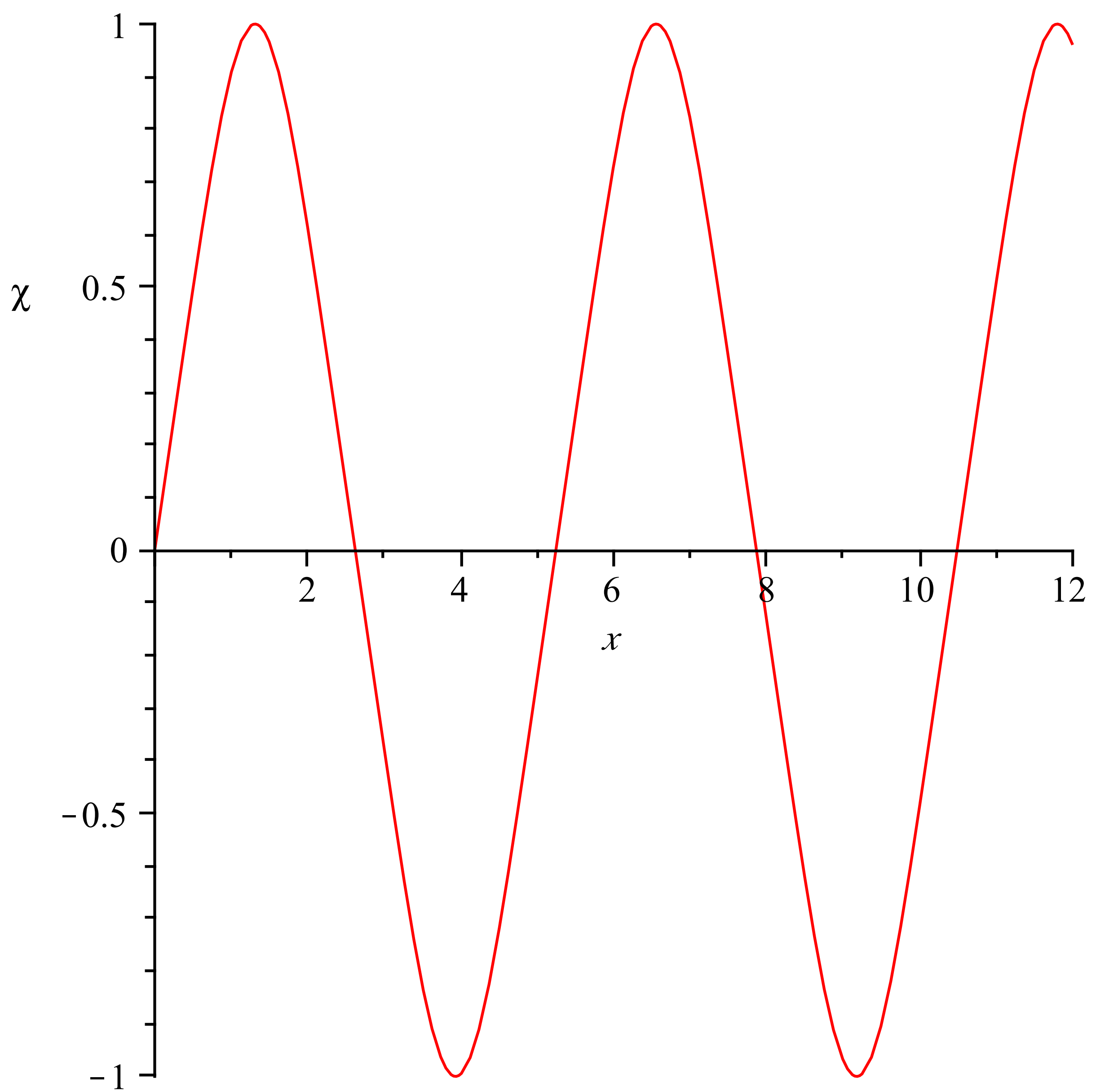
$$ds^2 = \frac{1}{\zeta^2}\, \mathrm{sn}^2\left(\sqrt{\frac{u}{\pi}}\, \frac{n_\mu \bar{x}^\mu}{4\zeta \ell_{\rm Pl}} + \alpha; i \right) \delta_{\mu\nu} d\bar{x}^\mu d\bar{x}^\nu$$

With this particular normalization of the coordinates, *the average of* ds^2 *equals the standard form of the line element on flat euclidean space,*

$$\langle ds^2 \rangle = \delta_{\mu\nu} d\bar{x}^\mu d\bar{x}^\nu ,$$

even though the scale factor is rapidly oscillating on short scales. If $u = O(1)$, the coordinate length $\Delta \bar{x}$ of one period is of the order of a Planck length.

sinus amplitudinis, solution of the Bogomolny equation



The variational approach

Let us consider:

$$S[\chi] = \frac{1}{2}\int d^4x\Big\{-(\partial_\mu\chi)^2 + \frac{1}{Q^2}\Big(\frac{\Box\chi}{\chi}\Big)^2\Big\}$$

where we abbreviated

$$Q \equiv (72\beta)^{-1/2}$$

and the test function:

$$\chi(z) = A\Big[1 + h\cos(\nu z)\Big]$$

so that:

$$S = \mathrm{Vol}\cdot\mathscr{S} \quad \text{with} \quad \mathscr{S} = \frac{\nu}{2\pi}\int_0^{2\pi/\nu} dz\Big\{-(\partial_\mu\chi)^2 + \frac{1}{Q^2}\Big(\frac{\Box\chi}{\chi}\Big)^2\Big\}$$

$$\mathscr{S} = \frac{\nu}{2\pi}\int_0^{2\pi/\nu} dz\,\frac{h^2\nu^2}{2Q^2\left(1 + 2\,h\,\cos(\nu\,z) + h^2\cos(\nu\,z)^2\right)} \tag{0.3}$$

$$\times\Big[-A^2Q^2 + A^2Q^2\cos(\nu\,z)^2 - 2\,A^2Q^2h\,\cos(\nu\,z) + 2\,A^2Q^2h\,\cos(\nu\,z)^3$$

$$-A^2Q^2h^2\cos(\nu\,z)^2 + A^2Q^2h^2\cos(\nu\,z)^4 + \cos(\nu\,z)^2\,\nu^2\Big]$$

The optimal trial metric

For the case $\Lambda = 0$ the absolute minimum is a *single* nonlinear plane wave which can be approximated by the harmonic ansatz

$$\chi(x) = A[1 + h\cos\left(\sqrt{\xi}\, QA\, n_\mu x^\mu + \alpha\right)]$$

with an arbitrary unit vector n_μ and phase α. The constants $h \approx 0.58$ and $\sqrt{\xi} = \nu/AQ \approx 0.45$ are universal numbers independent of β. Here it is natural to introduce

$$\bar{x}^\mu \equiv \sqrt{\frac{2\pi}{3}(2+h^2)}\, A\,\ell_{\rm Pl}\, x^\mu$$

so that

$$ds^2 = \frac{2}{2+h^2}\left[1 + h\cos\left(\sqrt{\frac{\xi}{48\pi(2+h^2)\beta}}\, m_{\rm Pl}\, n_\mu \bar{x}^\mu + \alpha\right)\right]^2 \delta_{\mu\nu} d\bar{x}^\mu d\bar{x}^\nu$$

Averaging over the harmonic oscillations we find a flat spacetime with $\langle ds^2 \rangle = \delta_{\mu\nu} d\bar{x}^\mu d\bar{x}^\nu$ again.

Analytical approximation and numerical investigation in D=1

Let us consider $x^1 \equiv x$ in the EOM:

$$\left[\sigma'' + (\sigma')^2\right] e^{2\sigma} + \frac{u}{6}\, e^{4\sigma} + 72\beta\left[\sigma'''' - 6\,\sigma''(\sigma')^2\right] = 0 \qquad (0.4)$$

Here the prime denotes a derivative with respect to x. If we expand we expand $e^\sigma = 1 + \sigma + \ldots$ we have

$$\sigma'' + \frac{u}{6}(1 + 4\sigma) + 72\beta\sigma'''' = 0$$

Its most general solution reads

$$\sigma(x) = -\frac{1}{4} + A_+ \cos(x\,\omega_+) + A_- \cos(x\,\omega_-) + B_+ \sin(x\,\omega_+) + B_- \sin(x\,\omega_-)$$

where $A_\pm$, $B_\pm$ are integration constants, and

$$\omega_\pm = \frac{1}{\sqrt{6}}\sqrt{Q(3Q \pm \sqrt{9Q^2 - 24u})}.$$

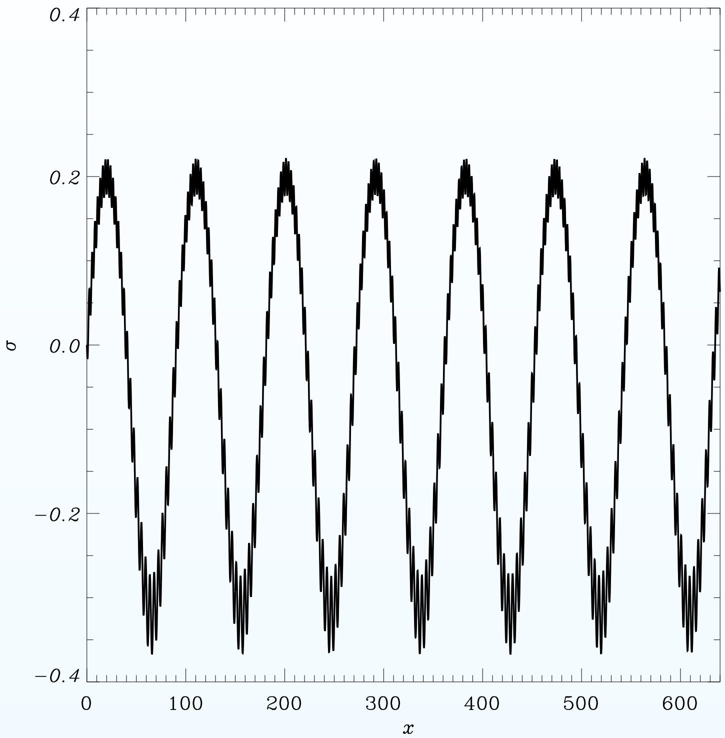
Analytical approximation and numerical investigation in D=1

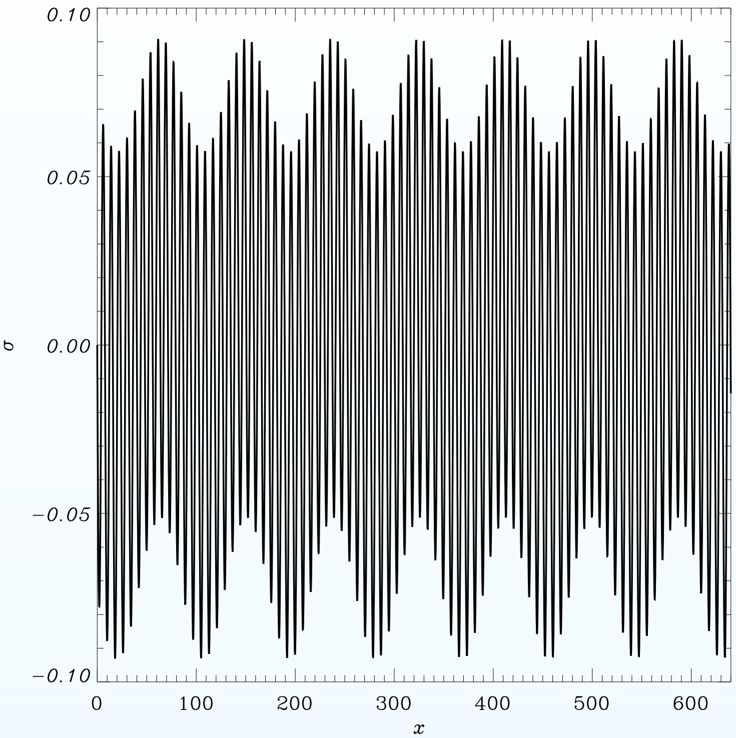
$$\omega_\pm = \frac{1}{\sqrt{6}}\sqrt{Q(3Q \pm \sqrt{9Q^2 - 24u})}.$$

$u \equiv \frac{16\pi}{3} G\Lambda$ and $Q^2 \equiv 1/(72\beta) > 0$. The constants $\omega_\pm$ are real, and so $\sigma(x)$ is periodic if $Q^2 \geq 8u/3$.

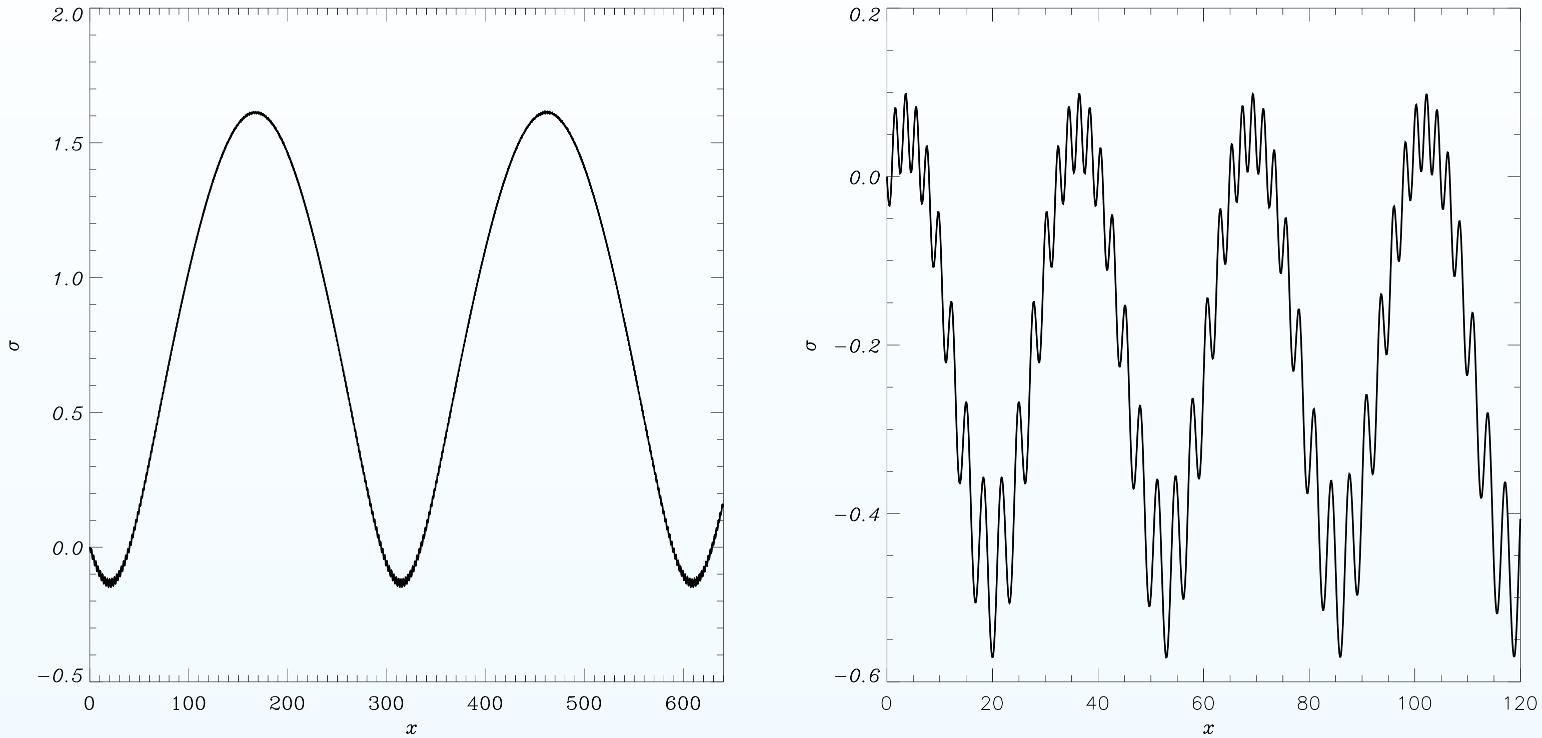
- For $(u = 0)$ we have $\omega_+ = Q$ and $\omega_- = 0$, whence $\sigma(x)$ is periodic with a single period $\propto 1/\beta$ determined by the R^2 term.
- When we switch on a small cosmological constant the “large" frequency $\omega_+ = Q$ is not affected much, but $\sigma(x)$ develops an additional periodicity with the “small" frequency ω_- and $\omega_- \approx \sqrt{2u/3} \propto \sqrt{\Lambda}$.
- In general the solution displays two different scales on which it varies. If $Q^2 \gg u$, $\omega_- \to 0$ and the only relevant frequency is the period of the R^2-term. For large Q-values the corresponding frequency is proportional to $1/\sqrt{\beta}$, and since $Qx \propto \bar{x}/(\sqrt{\beta}\ell_{\rm Pl})$ we see that dimensionful coordinate period $\Delta\bar{x}$ is of the order $\sqrt{\beta}\ell_{\rm Pl}$.

$u = 10^{-2}$ and $Q = 2$ (left) and $Q = 0.8$ (right)





$Q = 3$ and $u = 10^{-4}$ (left) and $u = 10^{-1}$



Lattice regulated model

$$
\begin{aligned}
S[\sigma(x)] = \sum_x \Big\{ \frac{u}{4!}\, e^{4\sigma(x)} \\
+ \sum_\mu \frac{1}{2} \Big[e^{2\sigma(x)} (\sigma(x+e_\mu) + \sigma(x-e_\mu) - 2\sigma(x) + (\sigma(x+e_\mu) - \sigma(x))^2) \\
+ \sum_\nu 36\beta(\sigma(x+e_\mu) + \sigma(x-e_\mu) - 2\sigma(x))(\sigma(x+e_\nu) + \sigma(x-e_\nu) - 2\sigma(x)) \\
+(\sigma(x+e_\mu) + \sigma(x-e_\mu) - 2\sigma(x))(\sigma(x+e_\nu) - \sigma(x)) \\
+(\sigma(x+e_\nu) + \sigma(x-e_\nu) - 2\sigma(x))(\sigma(x+e_\mu) - \sigma(x)) \\
+(\sigma(x+e_\mu) - \sigma(x))(\sigma(x+e_\nu) - \sigma(x))^2 \Big] \Big\}.
\end{aligned}
\tag{0.5}
$$

The variable σ is a dimensionless field so that the lattice cutoff is $a \equiv 1$, and ${e_\mu}^\nu = \delta_\mu^\nu$.

For computational reasons we work in $d = 2$ dimensions and we consider a two-dimensional lattice.

Lattice regulated model

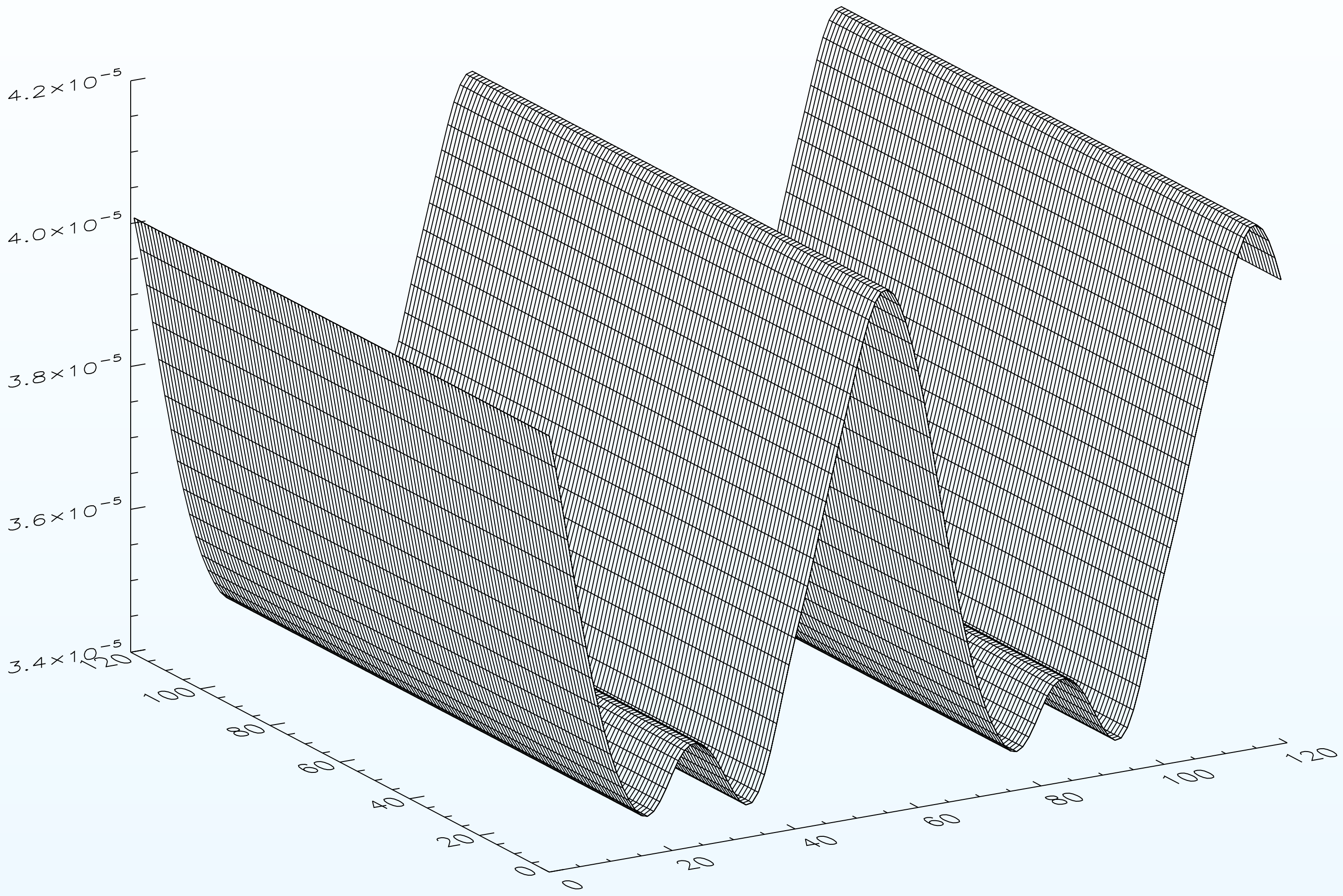


Figure 2: $\beta = 27$ and $u = 0$.

Lattice regulated D=2

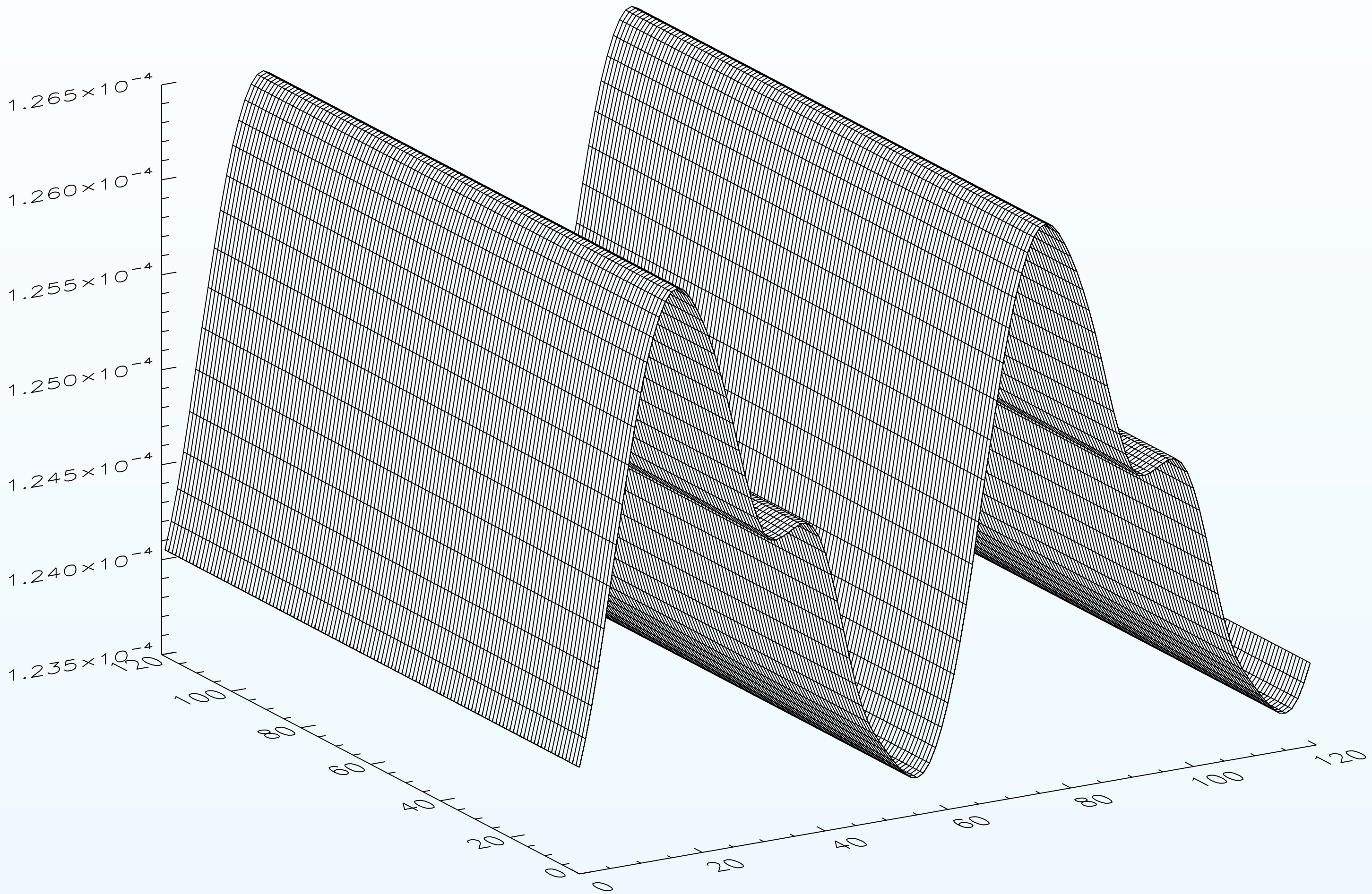


Figure 3: $\beta = 55$ and $u = 0$.

Deep connection with isotropic Lifshitz points in SM !

Critical behavior in a mixture of a Homopolymer blend and Diblock Copolimer (PRL, 82, 25, 5056):

$$H = \frac{1}{2} \int dx^3 \ c_2 (\nabla \phi)^2 + c_4 (\nabla^2 \phi)^2 + r\phi^2 + u\phi^4 + g\phi^6$$

A modulated phase is present if $c_2 < 0$ and a crossover between 3D Ising and isotropic lifshitz universality class is observed in three components mixture made of critical polymer blend.

Deep connection with isotropic Lifshitz points in SM !

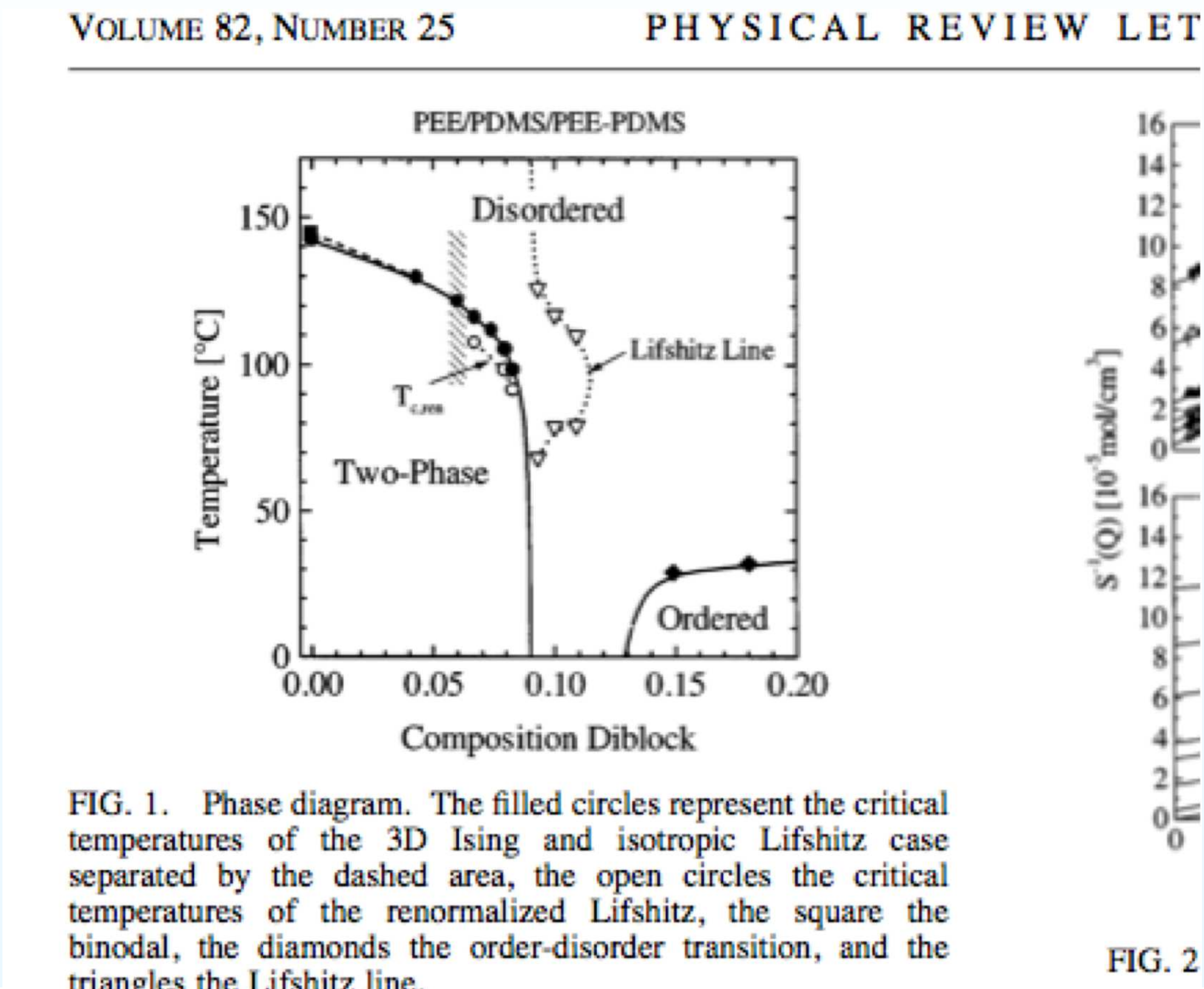


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FIG. 1. Phase diagram. The filled circles represent the critical temperatures of the 3D Ising and isotropic Lifshitz case separated by the dashed area, the open circles the critical temperatures of the renormalized Lifshitz, the square the binodal, the diamonds the order-disorder transition, and the triangles the Lifshitz line.

FIG. 2

Conclusions

- A stable minimum action configuration for QG is not flat space but sort of "lasagna vacuum" (not to be confused with "spaghetti vacuum" in QCD!)
- This "looks" flat over distances much larger than Planck size
- Lorentz symmetry is broken by the appearance of a minimum length scale. More information will arrive from an actual calculation of the propagator (work in progress with Martin)
- What is the role of the isotropic Lifshitz point in this picture?
- Understanding this phase can be of great "experimental" interest