Probing the quantum nature of spacetime by diffusion

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01/25– Dimensional flow in quantum gravity

- Perturbative QG: Renormalizable near $D = 2$ [Gastmans et al. 1978; Weinberg 1979; Kawai & Ninomiya 1990;. . .].
- String theory (worldsheet): conformal techniques.
- Noncommutative geometry: fundamental and effective level [Connes 2006; Benedetti 2008; Alesci & Arzano 2012].
- Spin foams: $d_S < 4$ [Modesto (et al.) 2008–10; G.C., Oriti & Thürigen to appear].
- Asymptotic safety (QEG): UV spectral dimension $d_S = 2$, intermediate regime $d_S \sim 4/3$ [Lauscher & Reuter 2005; Reuter & Saueressig 2011].
- CDT (simplicial path integral): $d_S \sim 2$ at small scales [Ambjørn et al. 2005; Benedetti & Henson 2009].
- HL gravity (effective QFT): UV $d_S = 2$ [Hořava 2008,2009].
- Multifractional spacetimes [G.C. 2009-2013].

02/25– How can we characterize effective quantum geometry?

- Dimensional flow: changing behaviour of correlation functions (as across a phase transition), spacetime with scale-dependent "dimension." Universal feature in quantum gravity related to UV finiteness ['t Hooft 1993; Carlip 2009,2010; G.C. 2009].
- *Matching d_S does not imply that the quantum spacetime found in different approaches is the same*. Spectral and Hausdorff dimension insufficient to fully characterize a geometry.
- To each (regime) of quantum geometry a stochastic process can be associated.
- Insight gained via the diffusion equation.

03/25– Diffusion, spectral and walk dimension

$$\left(\partial_\sigma - \nabla_x^2\right) P = 0\,, \qquad P(x,x',0) = \delta(x-x'),$$

$$P(x,x',\sigma) = P_1(x,x',\sigma) := \frac{\mathrm{e}^{-\frac{|x-x'|^2}{4\sigma}}}{(4\pi\sigma)^{D/2}} \geq 0.$$

Probabilistic interpretation: $P \geq 0$ is the PDF of Brownian motion.

Return probability $\mathcal{P}(\sigma) := \int \mathrm{d}^D x\, P(x,x,\sigma) \propto \sigma^{-D/2} = \sigma^{-d_S/2}$. In general,

$$d_S(\sigma) := -2\frac{\partial \ln \mathcal{P}(\sigma)}{\partial \ln \sigma}\,.$$

$$\langle X^2(\sigma)\rangle \propto \sigma^{2/d_W}\,, \qquad d_W = 2\frac{d_H}{d_S}$$

04/25– Where quantum gravity enters

$$\left(\partial_\sigma - \nabla_x^2\right) P = 0\,, \qquad P(x,x',0) = \delta(x-x').$$

An **anomalous** spectral dimension is determined by

- Diffusion operator $\partial_\sigma \to \sum_n \xi_n \partial_\sigma^{\beta_n}$ [QEG, HL, multiscale spacetimes].
- Laplacian $\nabla_x^2 \to \sum_n \zeta_n \mathcal{K}_{\gamma_n,\alpha_n}$ [QEG, HL, multiscale spacetimes].
- Initial condition $\delta(x-x') \to f(x,x')$ [noncomm. geometry].

Left Caputo derivative (useful for later):

$$(\partial^\beta f)(\sigma) := \frac{1}{\Gamma(1-\beta)} \int_0^\sigma \frac{\mathrm{d}\sigma'}{(\sigma-\sigma')^\beta} \partial_{\sigma'} f(\sigma')$$

05/25– Degeneracy problem in transport theory

$$\langle X^2(\sigma)\rangle \propto \sigma^\beta\,, \qquad \beta \leq 1 \qquad \text{(as in all QG models)}$$

Different ways to obtain subdiffusion [Sokolov 2012]:

- *Labyrinthine* diffusion (on fractals), probe meets obstacles and dead ends ("rough landscape").
  $\{\partial_\sigma^\beta - \partial_x[A(x)\partial_x B(x)]\}P = 0$.
- *Continuous time random walk* (CTRW), probe trapped in bound states where it spends more time than in free motion ("flat valleys surrounded by high ridges").
  $(\partial_\sigma - \partial_\sigma^{1-\beta}\nabla_x^2)P = 0$, $(\partial_\sigma^\beta - \nabla_x^2)P = 0$.
- *Fractional Brownian motion* (FBM), probe dragged by complex environment (viscoelastic behavior).
- *Scaled Brownian motion* (SBM), used as a fitting model of data with anomalous scaling (e.g., in biophysics).

06/25– Improving our understanding of effective quantum geometries

- $P \geq 0$ criterion to fix the diffusion equation in QG and necessary condition to get meaningful d_S.
- Study of $P(x, x', \sigma)$ can disentangle the degeneracy in QG models and better characterize their geometry.
- One stochastic process for each geometry?

Outline

07/25– Scalings

Reuter, Bonanno, Lauscher, Litim, Saueressig, . . .

• Assuming all dimensionless couplings approach a UV fixed point implies that the scale-dependent average metric $\langle g_{\mu\nu}\rangle_k \propto k^{-2}$ as $k \to \infty$.
• Cosmological constant $\bar{\lambda}_k$ also running with the scale:

$$\bar{\lambda}_k = F(k^2)\bar{\lambda}_{k_0}\,.$$

In the absence of matter and in the Einstein–Hilbert truncation of the effective action Γ_k,

$$\langle g^{\mu\nu}\rangle_k = F(k^2)\langle g^{\mu\nu}\rangle_{k_0}\,.$$

There follows the scaling of the generalized Laplacian:

$$\Delta(k) = F(k^2)\,\Delta(k_0)\,.$$

F determined by RG trajectory.

08/25– Previous Ansatz for the diffusion equation

Lauscher & Reuter 2005

$$(\partial_\sigma - \nabla_x^2)P = 0 \qquad \to \qquad (\partial_\sigma - \langle\Delta_x[g]\rangle)P = 0$$

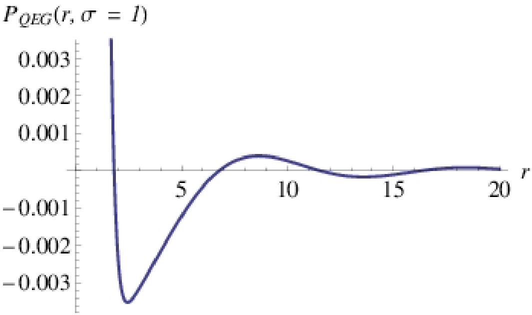
Solution in improvement schemes ($k = p$ physical momentum):

$$P_{\mathrm{QEG}}(x,x',\sigma) = \int \frac{\mathrm{d}^D p}{(2\pi)^D}\, \mathrm{e}^{\mathrm{i}p\cdot(x-x')}\, \mathrm{e}^{-\sigma\, p^2\, F(p^2)}\,.$$

In asymptotic regimes, $F \sim |k|^\delta$ and

$$d_{\mathrm{S}} = \frac{2D}{2+\delta}\,, \qquad \delta = 0\ (\mathrm{IR}),\ 4\ (\mathrm{semiclassical}),\ 2\ (\mathrm{UV})$$

09/25– Probabilistic interpretation fails



Task: find an alternative diffusion equation for QEG such that $P_{\mathrm{QEG}} \geq 0$ and the *same* d_S is recovered.

Outline

10/25– Case 1: Nonlinear time

$F \sim k^\delta$:

$$\left(\partial_\sigma - k^\delta \langle g^{\mu\nu}\nabla_\mu\nabla_\nu\rangle_{k_0}\right) P = 0 \qquad \Rightarrow \qquad \left(k^{-\delta}\partial_\sigma - \nabla_x^2\right) P = 0$$

Scale identification $k = k(\sigma)$. Dimensionally, $\sigma \sim k^{-\delta-2}$ and thus

$$\left(\frac{\partial}{\partial\sigma^\beta} - \nabla_x^2\right) P(x, x', \sigma) = 0\,, \qquad \beta = \frac{2}{\delta+2}$$

- Positive solution, $P = (4\pi\sigma^\beta)^{-D/2} \mathrm{e}^{-\frac{r^2}{4\sigma^\beta}} > 0$.
- Same $d_S = D\beta$ as before.
- Diffusion process determined univocally? No, twin problem: either FBM or SBM.

11/25– Fractional BM

[Barnes & Allan 1966; Mandelbrot & Van Ness 1968; Metzler & Klafter 2004]

- Langevin equation:

$$\begin{aligned}\partial_\sigma X_{\mathrm{FBM}}(\sigma) &= \partial_\sigma[{}_{-\infty}I^{\frac{\beta+1}{2}}\eta](\sigma)\\ &:= \partial_\sigma \int_{-\infty}^{\sigma} d\sigma' \, \frac{(\sigma-\sigma')^{\frac{\beta-1}{2}}}{\Gamma(\gamma)}\,\eta(\sigma')\,,\end{aligned}$$

- Stationary increments:
$\langle[X_{\mathrm{FBM}}(\sigma)-X_{\mathrm{FBM}}(\sigma')]^2\rangle = \langle X^2_{\mathrm{FBM}}(\sigma-\sigma')\rangle$.
- Non-Markovian: defined as a nonlocal process with dependence on past history.
- Variance $\langle X^2(\sigma)\rangle \propto \sigma^\beta$ (remember that $d_{\mathrm{W}} = 2d_{\mathrm{H}}/d_{\mathrm{S}} = 2/\beta$).

12/25– Scaled BM

[Lim & Muniandy 2002; Sokolov 2012]

- Brownian motion with power-law time: $X_{\mathrm{SBM}}(\sigma) := B(\sigma^\beta)$. Langevin equation

$$\partial_\sigma X_{\mathrm{SBM}}(\sigma) = \sigma^{\frac{\beta-1}{2}}\eta(\sigma)\,.$$

- Non-stationary increments: $\langle [X_{\mathrm{SBM}}(\sigma) - X_{\mathrm{SBM}}(\sigma')]^2\rangle \neq \langle X^2_{\mathrm{SBM}}(\sigma-\sigma')\rangle$.
- Markovian: scale transformation $\sigma \to \sigma^\beta$ preserves time ordering for $\beta > 0$.
- Variance $\langle X^2(\sigma)\rangle \propto \sigma^\beta$.

13/25– Qualitative interpretation

- Neither FBM nor SBM strictly correspond to diffusion on a fractal. Notion of a “fractal spacetime” not supported mathematically in this case, even if spacetime dimension is anomalous.
- Subdiffusion occurs because of a viscoelastic effect: test particle slowed down by quantum fluctuations of spacetime.

14/25– Case 2: Fractional diffusion

$F \sim k^\delta$:

$$\left(k^{-\delta}\partial_\sigma - \nabla_x^2\right) P = 0$$

Laplace transform:

$$k^{-\delta}(s)\left\{s\mathcal{L}[P(x,x',\sigma)] - P(x,x',0)\right\} - \nabla_x^2 P(x,x',\sigma) = 0$$

Scale identification $k = k(s) = s^{\frac{1}{\delta+2}}$.
Comparing with the Laplace transform of the Caputo derivative,

$$\mathcal{L}[\partial^\beta f(\sigma)](s) = s^\beta \mathcal{L}[f(\sigma)] - s^{\beta-1} f(0)\,,$$

we get

$$\boxed{\left(\partial_\sigma^\beta - \nabla_x^2\right) P_\beta(x,x',\sigma) = 0}$$

14/25– Case 2: Solution

$$P_\beta(r,\sigma) = \int_0^\infty ds\, A_\beta(s,\sigma)\, P_1(r,s)\,,$$

$$A_\beta = \frac{1}{\beta\, s} H^{1,0}_{1,1}\left[\frac{s^{1/\beta}}{\sigma}\,\middle|\, \begin{matrix}(1,1)\\ (1,1/\beta)\end{matrix}\right] = \frac{1}{s}\sum_{n=0}^\infty \frac{(-1)^n}{\Gamma(1-\beta-\beta n)\Gamma(1+n)}\left(\frac{s}{\sigma^\beta}\right)^{1+n}.$$

Stochastic process for $\beta = 1/n$: multiply iterated Brownian motion (IBM) [Burdzy 1992; Orsingher & Beghin 2004-2009], a special case of CTRW: $X_{\rm IBM}(\sigma) = B_1[|B_2[\ldots|B_n(\sigma)|]|]$.

15/25– Case 2: Remarks

- Other fractional derivatives (Riemann–Liouville, ...) excluded; with Caputo, standard Cauchy problem.
- Positive solution, $P > 0$.
- Same $d_S = D\beta$ as before.
- Diffusion process non-Gaussian and determined univocally: IBM.
- Physical interpretation (stems from known application of IBM): quantum-gravity effects turn a smooth background into an effective spacetime where diffusing particles are subjected to a randomized diffusion time, and diffuse as if in a "crack" formed by quantum fluctuations of the geometry.
- Can be distinguished from Case 1 by looking at higher moments.

16/25– Comparison: BM, F/SBM, IBM

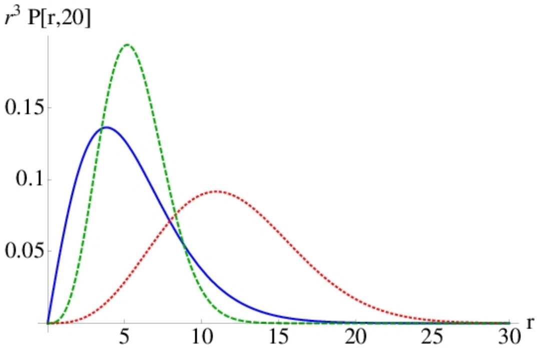


Figure : PDF for normal diffusion (red dotted line), power-law diffusion with $\beta = 1/2$ (green dashed line) and fractional diffusion with $\beta = 1/2$ (thick blue line)

Outline

17/25– Anisotropic scalings

Hořava 2009, . . .

• Coordinates *defined* to have anomalous scaling:

$$t \to \lambda^z t\,, \qquad \vec{x} \to \lambda \vec{x}$$

as in multicritical systems [Lifshitz 1941].

• Naturally reproducible in multifractional spacetimes when the above coordinates are regarded as composite objects [G.C. arXiv:1209.4376].

• Action invariant under time reparametrizations and diffeomorphisms of spatial slices. Laplace–Beltrami operator in the UV replaced by $\Box \to -\partial_t^2 + (\nabla_x^2)^z$.

• Quantum effects encoded in a modified dispersion relation $\omega^2 = f(\vec{p}^2)$, where $f \propto (\vec{p}^2)^z$ in the UV and $f \propto \vec{p}^2$ in the IR.

18/25– Previous Ansatz for the diffusion equation

Hořava 2009

In the UV ($z = D - 1$):

$$\left[\partial_\sigma - \partial_t^2 - (\nabla_x^2)^z\right] P_{\rm HL}(r, t, \sigma) = 0$$

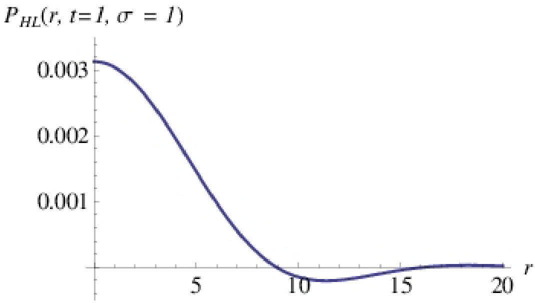
Solution is

$$P_{\rm HL}(x, x', \sigma) = \int \frac{{\rm d}^2 p\, {\rm d}\omega}{(2\pi)^3}\, {\rm e}^{{\rm i}\vec{p}\cdot(\vec{x}-\vec{x}')} {\rm e}^{{\rm i}\omega t}\, {\rm e}^{-\sigma\,[\omega^2 + f(\vec{p}^2)]}$$

Smooth profile for f can be determined through a comparison with CDT data [Benedetti & Henson 2009; Sotiriou et al. 2011].

$$d_{\rm S} = 1 + \frac{D-1}{z} = D \text{ (IR)},\ 2 \text{ (UV)}$$

19/25– Probabilistic interpretation fails



Task: find an alternative diffusion equation for HL gravity such that $P_{\text{HL}} \geq 0$ and the *same* d_S is recovered.

Outline

20/25– An interesting duality

The PDF solving $(\partial_\sigma^\beta - \nabla_x^2)P_\beta = 0$ can also be obtained as a solution of an ordinary partial differential equation including higher-order Laplacians and source terms [Allouba & Zheng 2001; Baumer et al. 2009; Nane 2010].

For example, for $\beta = 1/2$ the diffusion equation dual to $(\partial_\sigma^{1/2} - \partial_x^2)P_{1/2} = 0$ is

$$(\partial_\sigma - \partial_x^4)P_{1/2}(r,\sigma) = \mathcal{S}_{1/2}\,, \qquad \mathcal{S}_{1/2} = \frac{1}{\sqrt{\pi\sigma}}\,\partial_x^2\,P_{1/2}(r,0)\,.$$

21/25– HL diffusion equation

The above duality suggests to look for a higher-order diffusion equation with spatial source terms, $\beta \leftrightarrow 1/z$:

$$\left[\partial_\sigma - \partial_t^2 - (\nabla_x^2)^z\right] P_{\rm HL}(t,r,\sigma) = P_1(t,\sigma)\,\mathcal{S}_{1/z}(r,\sigma)\,.$$

For $z = 3$:

$$\mathcal{S}_{1/3} = \frac{1}{\sigma^{1/3}\,\Gamma(2/3)}\,(\nabla_x^2)^2 P_{1/3}(r,0) + \frac{1}{\sigma^{2/3}\,\Gamma(1/3)}\,\nabla_x^2 P_{1/3}(r,0)\,.$$

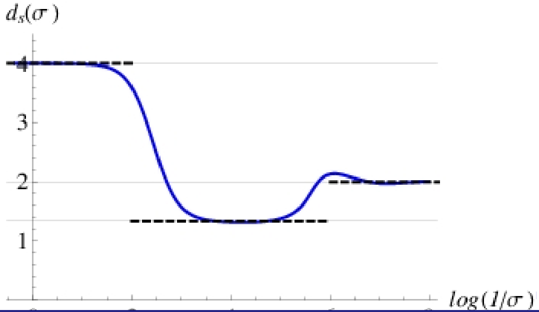
Solution

$$P_{\rm HL}(t,r,\sigma) = P_1(t,\sigma)\,P_{1/z}(r,\sigma) \geq 0\,.$$

Manifestly positive solution and same $d_{\rm S} = 1 + (D-1)/z$ as before!

22/25– Full dimensional flow: QEG

QEG: Generalizing $\delta \to \delta(\sigma)$ is similar to what is sometimes done in transport theory (time-dependent anomalous exponent [Lim & Muniandy 2000,2002; Zaslavsky 2002]. $\delta(k)$ determined numerically for a given RG trajectory [Reuter & Saueressig 2011,2012; Rechenberger & Saueressig 2012].



23/25– Monotonic flow: a stochastic analytic example

Brownian-time telegraph process [Orsingher & Beghin 2009]:

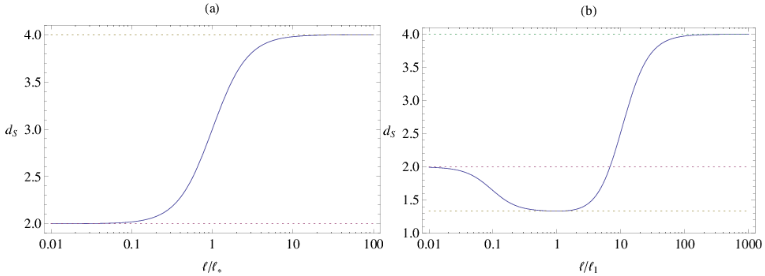
$$\left(\partial_\sigma + \frac{1}{\ell_1}\,\partial_\sigma^{\frac{1}{2}} - \nabla_x^2\right) P = 0\,,$$

$$X(\sigma) = T[|B(\sigma)|]\,, \qquad T(t) = \int_0^t \mathrm{d}s\,(-1)^{\mathcal{N}(s)}.$$

Kink-like spectral dimension.

24/25– Full dimensional flow: multifractional spacetimes

The above plot, the monotonic flow and other cases can be reproduced in multifractional spacetimes when the action measure has a hierarchy of length scales [G.C. 2011-2013]. Analytic control.



25/25– Conclusions

- QEG and HL diffusion equations with probabilistic interpretation (well-defined d_S).
- More precise characterization of quantum spacetimes by their stochastic properties now possible.
- No unique Ansatz for the QEG diffusion equation.
- Twin problem for one of the QEG Ansätze.
- Interpretation of source term in HL diffusion equation still unclear.
- Multiscale cases under analytic control but to be developed in more detail.
- Stimulates **comparison with the solution P and higher moments** of other approaches such as CDT.

Thank you!