Probing the quantum nature of spacetime by diffusion

based on arXiv:1304.7247

(in collaboration with A. Eichhorn and F. Saueressig)

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Dimensional flow in quantum gravity

- **Perturbative QG**: Renormalizable near $D = 2$ [Gastmans et al. 1978; Weinberg 1979; Kawai & Ninomiya 1990; ...].

- **String theory** *(worldsheet)*: conformal techniques.

- **Noncommutative geometry**: fundamental and effective level [Connes 2006; Benedetti 2008; Alesci & Arzano 2012].

- **Spin foams**: $d_S < 4$ [Modesto (et al.) 2008–10; G.C., Oriti & Thürigen to appear].

- **Asymptotic safety** *(QEG)*: UV **spectral dimension** $d_S = 2$, intermediate regime $d_S \sim 4/3$ [Lauscher & Reuter 2005; Reuter & Saueressig 2011].

- **CDT** *(simplicial path integral)*: $d_S \sim 2$ at small scales [Ambjørn et al. 2005; Benedetti & Henson 2009].

- **HL gravity** *(effective QFT)*: UV $d_S = 2$ [Hořava 2008, 2009].

- **Multifractional spacetimes** [G.C. 2009-2013].
How can we characterize effective quantum geometry?

- Dimensional flow: changing behaviour of correlation functions (as across a phase transition), spacetime with scale-dependent “dimension.” Universal feature in quantum gravity related to UV finiteness [’t Hooft 1993; Carlip 2009,2010; G.C. 2009].

- **Matching** $d_S$ **does not imply that the quantum spacetime found in different approaches is the same.** Spectral and Hausdorff dimension **insufficient** to fully characterize a geometry.

- To each (regime) of quantum geometry a **stochastic process** can be associated.

- Insight gained via the **diffusion equation**.

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Probing the quantum nature of spacetime by diffusion ([arXiv:1304.7247](https://arxiv.org/abs/1304.7247))
Diffusion, spectral and walk dimension

\[
(\partial_\sigma - \nabla^2_x) P = 0, \quad P(x, x', 0) = \delta(x - x'),
\]

\[
P(x, x', \sigma) = P_1(x, x', \sigma) := e^{-\frac{|x-x'|^2}{4\sigma}} \frac{1}{(4\pi \sigma)^{D/2}} \geq 0.
\]

Probabilistic interpretation: \( P \geq 0 \) is the PDF of Brownian motion.

Return probability \( \mathcal{P}(\sigma) := \int d^Dx \, P(x, x, \sigma) \propto \sigma^{-D/2} = \sigma^{-d_S/2} \). In general,

\[
d_S(\sigma) := -2 \frac{\partial \ln \mathcal{P}(\sigma)}{\partial \ln \sigma}.
\]

\[
\langle X^2(\sigma) \rangle \propto \sigma^{2/d_W}, \quad d_W = 2 \frac{d_H}{d_S}
\]
Where quantum gravity enters

\[(\partial_\sigma - \nabla_x^2) P = 0, \quad P(x, x', 0) = \delta(x - x').\]

An **anomalous** spectral dimension is determined by

- Diffusion operator \(\partial_\sigma \rightarrow \sum_n \xi_n \partial_\sigma^{\beta_n}\) [QEG, HL, multiscale spacetimes].
- Laplacian \(\nabla_x^2 \rightarrow \sum_n \zeta_n K_{\gamma_n, \alpha_n}\) [QEG, HL, multiscale spacetimes].
- Initial condition \(\delta(x - x') \rightarrow f(x, x')\) [noncomm. geometry].

Left Caputo derivative (useful for later):

\[(\partial^\beta f)(\sigma) := \frac{1}{\Gamma(1 - \beta)} \int_0^\sigma \frac{d\sigma'}{(\sigma - \sigma')^\beta} \partial_\sigma' f(\sigma').\]
Degeneracy problem in transport theory

\[ \langle X^2(\sigma) \rangle \propto \sigma^\beta, \quad \beta \leq 1 \] (as in all QG models)

Different ways to obtain subdiffusion [Sokolov 2012]:

- **Labyrinthine** diffusion (on fractals), probe meets obstacles and dead ends (“rough landscape”).
  \[ \{ \partial_\sigma^\beta - \partial_x[A(x)\partial_xB(x)] \} P = 0. \]

- **Continuous time random walk** (CTRW), probe trapped in bound states where it spends more time than in free motion (“flat valleys surrounded by high ridges”).
  \[ (\partial_\sigma - \partial_\sigma^{1-\beta} \nabla_x^2)P = 0, \quad (\partial_\sigma^\beta - \nabla_x^2)P = 0. \]

- **Fractional Brownian motion** (FBM), probe dragged by complex environment (viscoelastic behavior).

- **Scaled Brownian motion** (SBM), used as a fitting model of data with anomalous scaling (e.g., in biophysics).
Improving our understanding of effective quantum geometries

- $P \geq 0$ criterion to fix the diffusion equation in QG and necessary condition to get meaningful $d_S$.
- Study of $P(x, x', \sigma)$ can disentangle the degeneracy in QG models and better characterize their geometry.
- One stochastic process for each geometry?
Outline

1 Motivation

2 Asymptotic safety
   - Review and problem
   - Novel RG-improved diffusion equations

3 Hořava–Lifshitz spacetimes
   - Review and problem
   - Novel diffusion equation

4 Multiscale geometries
07/25– **Scalings**

Reuter, Bonanno, Lauscher, Litim, Saueressig, . . .

- Assuming all dimensionless couplings approach a UV fixed point implies that the scale-dependent average metric
  \[ \langle g_{\mu\nu} \rangle_k \propto k^{-2} \text{ as } k \to \infty. \]

- Cosmological constant \( \bar{\lambda}_k \) also running with the scale:
  \[ \bar{\lambda}_k = F(k^2)\bar{\lambda}_0. \]

In the absence of matter and in the Einstein–Hilbert truncation of the effective action \( \Gamma_k \),

\[ \langle g^{\mu\nu} \rangle_k = F(k^2)\langle g^{\mu\nu} \rangle_0. \]

There follows the scaling of the generalized Laplacian:

\[ \Delta(k) = F(k^2) \Delta(k_0). \]

\( F \) determined by RG trajectory.
Previous Ansatz for the diffusion equation
Lauscher & Reuter 2005

\[(\partial_\sigma - \nabla_\sigma^2)P = 0 \quad \rightarrow \quad (\partial_\sigma - \langle \Delta_x[g] \rangle)P = 0\]

Solution in improvement schemes \((k = p \text{ physical momentum}):\)

\[P_{\text{QEG}}(x, x', \sigma) = \int \frac{d^Dp}{(2\pi)^D} e^{ip \cdot (x-x')} e^{-\sigma p^2 F(p^2)}.\]

In asymptotic regimes, \(F \sim |k|^{\delta}\) and

\[d_s = \frac{2D}{2 + \delta}, \quad \delta = 0 \text{ (IR)}, 4 \text{ (semiclassical)}, 2 \text{ (UV)}\]
Probabilistic interpretation fails

Task: find an alternative diffusion equation for QEG such that $P_{QEG} \geq 0$ and the same $d_S$ is recovered.
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Probing the quantum nature of spacetime by diffusion (arXiv:1304.7247)
Case 1: Nonlinear time

\[ F \sim k^\delta : \]

\[ \left( \partial_\sigma - k^\delta \left( g^{\mu\nu} \nabla_\mu \nabla_\nu \right)_{k_0} \right) P = 0 \quad \Rightarrow \quad \left( k^{-\delta} \partial_\sigma - \nabla_x^2 \right) P = 0 \]

Scale identification \( k = k(\sigma) \). Dimensionally, \( \sigma \sim k^{-\delta-2} \) and thus

\[ \left( \frac{\partial}{\partial \sigma \beta} - \nabla_x^2 \right) P(x, x', \sigma) = 0, \quad \beta = \frac{2}{\delta + 2} \]

- Positive solution, \( P = (4\pi \sigma^\beta)^{-D/2} e^{-\frac{r^2}{4\sigma^\beta}} > 0 \).
- Same \( d_S = D\beta \) as before.
- Diffusion process determined univocally? No, twin problem: either FBM or SBM.
Fractional BM

[Barnes & Allan 1966; Mandelbrot & Van Ness 1968; Metzler & Klafter 2004]

- Langevin equation:
  \[ \partial_\sigma X_{\text{FBM}}(\sigma) = \partial_\sigma \left[ -\infty I^{\beta+1/2} \eta(\sigma) \right] \]
  \[ := \partial_\sigma \int_{-\infty}^\sigma d\sigma' \left( \frac{\sigma - \sigma'}{2} \right)^{\beta-1/2} \Gamma(\gamma) \eta(\sigma') , \]

- Stationary increments:
  \[ \langle [X_{\text{FBM}}(\sigma) - X_{\text{FBM}}(\sigma')]^2 \rangle = \langle X_{\text{FBM}}^2(\sigma - \sigma') \rangle . \]

- Non-Markovian: defined as a nonlocal process with dependence on past history.

- Variance \[ \langle X^2(\sigma) \rangle \propto \sigma^\beta \] (remember that \[ d_W = 2d_H/d_S = 2/\beta \]).
Brownian motion with power-law time: $X_{SBM}(\sigma) := B(\sigma^\beta)$.

Langevin equation

$$\partial_\sigma X_{SBM}(\sigma) = \sigma^{\frac{\beta-1}{2}} \eta(\sigma).$$

Non-stationary increments:

$$\langle [X_{SBM}(\sigma) - X_{SBM}(\sigma')]^2 \rangle \neq \langle X_{SBM}^2(\sigma - \sigma') \rangle.$$ 

Markovian: scale transformation $\sigma \to \sigma^\beta$ preserves time ordering for $\beta > 0$.

Variance $\langle X^2(\sigma) \rangle \propto \sigma^\beta$. 

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Neither FBM nor SBM strictly correspond to diffusion on a fractal. Notion of a “fractal spacetime” not supported mathematically in this case, even if spacetime dimension is anomalous.

Subdiffusion occurs because of a viscoelastic effect: test particle slowed down by quantum fluctuations of spacetime.
Motivation
Asymptotic safety
Horava–Lifshitz spacetimes
Multiscale geometries

Novel RG-improved diffusion equations

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Case 2: Fractional diffusion

\[ F \sim k^\delta : \quad \left( k^{-\delta} \partial_\sigma - \nabla^2_x \right) P = 0 \]

Laplace transform:

\[ k^{-\delta}(s) \left\{ s \mathcal{L}[P(x, x', \sigma)] - P(x, x', 0) \right\} - \nabla^2_x P(x, x', \sigma) = 0 \]

Scale identification \( k = k(s) = s^{\frac{1}{\delta+2}} \).

Comparing with the Laplace transform of the Caputo derivative,

\[ \mathcal{L}[\partial^\beta f(\sigma)](s) = s^\beta \mathcal{L}[f(\sigma)] - s^{\beta-1}f(0), \]

we get

\[ \left( \partial_\sigma^\beta - \nabla^2_x \right) P_\beta(x, x', \sigma) = 0 \]
Case 2: Solution

\[ P_\beta(r, \sigma) = \int_0^\infty ds \, A_\beta(s, \sigma) \, P_1(r, s), \]

\[ A_\beta = \frac{1}{\beta s} H_{1,1}^{1,0} \left[ \frac{s^{1/\beta}}{\sigma} \left| \begin{array}{c} (1, 1) \\ (1, 1/\beta) \end{array} \right. \right] = \frac{1}{s} \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(1-\beta-\beta n)\Gamma(1+n)} \left( \frac{s}{\sigma^\beta} \right)^{1+n}. \]

Stochastic process for \( \beta = 1/n \): multiply iterated Brownian motion (IBM) [Burdzy 1992; Orsingher & Beghin 2004-2009], a special case of CTRW: \( X_{\text{IBM}}(\sigma) = B_1[B_2[\ldots B_n(\sigma)]]]. \)
Case 2: Remarks

- Other fractional derivatives (Riemann–Liouville, ...) excluded; with Caputo, standard Cauchy problem.
- Positive solution, $P > 0$.
- Same $d_S = D\beta$ as before.
- Diffusion process non-Gaussian and determined univocally: IBM.
- Physical interpretation (stems from known application of IBM): quantum-gravity effects turn a smooth background into an effective spacetime where diffusing particles are subjected to a randomized diffusion time, and diffuse as if in a “crack” formed by quantum fluctuations of the geometry.
- Can be distinguished from Case 1 by looking at higher moments.
Motivation

Asymptotic safety

Hořava–Lifshitz spacetimes

Multiscale geometries

Novel RG-improved diffusion equations

Comparison: BM, F/SBM, IBM

Figure: PDF for normal diffusion (red dotted line), power-law diffusion with $\beta = 1/2$ (green dashed line) and fractional diffusion with $\beta = 1/2$ (thick blue line)
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Anisotropic scalings

Hořava 2009, ... 

- Coordinates *defined* to have anomalous scaling:
  \[ t \rightarrow \lambda^z t, \quad \vec{x} \rightarrow \lambda \vec{x} \]

as in multicritical systems [Lifshitz 1941].

- Naturally reproducible in multifractional spacetimes when the above coordinates are regarded as composite objects [G.C. arXiv:1209.4376].

- Action invariant under time reparametrizations and diffeomorphisms of spatial slices. Laplace–Beltrami operator in the UV replaced by \[ \Box \rightarrow -\partial_t^2 + (\nabla_{\vec{x}}^2)^z. \]

- Quantum effects encoded in a modified dispersion relation \[ \omega^2 = f(\vec{p}^2), \quad \text{where } f \propto (\vec{p}^2)^z \text{ in the UV and } f \propto \vec{p}^2 \text{ in the IR.} \]
Previous Ansatz for the diffusion equation

Hořava 2009

In the UV ($z = D - 1$):

$$\left[ \partial_\sigma - \partial_t^2 - (\nabla_x^2)^z \right] P_{\text{HL}}(r, t, \sigma) = 0$$

Solution is

$$P_{\text{HL}}(x, x', \sigma) = \int \frac{d^2p \, d\omega}{(2\pi)^3} \, e^{i\vec{p} \cdot (\vec{x} - \vec{x}')} \, e^{i\omega t} \, e^{-\sigma [\omega^2 + f(\vec{p}^2)]}$$

Smooth profile for $f$ can be determined through a comparison with CDT data [Benedetti & Henson 2009; Sotiriou et al. 2011].

$$d_s = 1 + \frac{D - 1}{z} = D \text{ (IR)}, \ 2 \text{ (UV)}$$
Probabilistic interpretation fails

Task: find an alternative diffusion equation for HL gravity such that $P_{\text{HL}} \geq 0$ and the same $d_S$ is recovered.
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Novel diffusion equation

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An interesting duality

The PDF solving \((\partial_\sigma^\beta - \nabla_x^2)P_\beta = 0\) can also be obtained as a solution of an ordinary partial differential equation including higher-order Laplacians and source terms [Allouba & Zheng 2001; Baumer et al. 2009; Nane 2010].

For example, for \(\beta = 1/2\) the diffusion equation dual to \((\partial_\sigma^{1/2} - \partial_x^2)P_{1/2} = 0\) is

\[
(\partial_\sigma - \partial_x^4)P_{1/2}(r, \sigma) = S_{1/2}, \quad S_{1/2} = \frac{1}{\sqrt{\pi \sigma}} \partial_x^2 P_{1/2}(r, 0).
\]
The above duality suggests to look for a higher-order diffusion equation with spatial source terms, $\beta \leftrightarrow 1/z$:

$$\left[ \partial_\sigma - \partial_t^2 - (\nabla^2_x)^z \right] P_{\text{HL}}(t, r, \sigma) = P_1(t, \sigma) S_{1/z}(r, \sigma).$$

For $z = 3$:

$$S_{1/3} = \frac{1}{\sigma^{1/3} \Gamma(2/3)} (\nabla^2_x)^2 P_{1/3}(r, 0) + \frac{1}{\sigma^{2/3} \Gamma(1/3)} \nabla^2_x P_{1/3}(r, 0).$$

Solution

$$P_{\text{HL}}(t, r, \sigma) = P_1(t, \sigma) P_{1/z}(r, \sigma) \geq 0.$$
Full dimensional flow: QEG

QEG: Generalizing $\delta \rightarrow \delta(\sigma)$ is similar to what is sometimes done in transport theory (time-dependent anomalous exponent [Lim & Muniandy 2000,2002; Zaslavsky 2002]. $\delta(k)$ determined numerically for a given RG trajectory [Reuter & Saueressig 2011,2012; Rechenberger & Saueressig 2012].
Monotonic flow: a stochastic analytic example

Brownian-time telegraph process [Orsingher & Beghin 2009]:

\[
\left( \partial_\sigma + \frac{1}{\ell_1} \partial_{\sigma}^{1/2} - \nabla_x^2 \right) P = 0 ,
\]

\[
X(\sigma) = T[|B(\sigma)|] , \quad T(t) = \int_0^t ds \ (-1)^{\mathcal{N}(s)} .
\]

Kink-like spectral dimension.
Full dimensional flow: multifractional spacetimes

The above plot, the monotonic flow and other cases can be reproduced in multifractional spacetimes when the action measure has a hierarchy of length scales [G.C. 2011-2013]. Analytic control.
Conclusions

- QEG and HL diffusion equations with probabilistic interpretation (well-defined $d_S$).
- More precise characterization of quantum spacetimes by their stochastic properties now possible.
- No unique Ansatz for the QEG diffusion equation.
- Twin problem for one of the QEG Ansätze.
- Interpretation of source term in HL diffusion equation still unclear.
- Multiscale cases under analytic control but to be developed in more detail.
- Stimulates *comparison with the solution* $P$ and higher moments of other approaches such as CDT.

Thank you!