Perturbation theory and the functional renormalization group.

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With M.Demmel and A.Codello

We usually start from the path-integral

$$\mathrm{e}^{-\Gamma_k[\phi]} = \int \mathcal{D}\varphi \, \mathrm{e}^{-S[\varphi] + \frac{\delta \Gamma_k}{\delta \phi}(\phi - \varphi) - \Delta S_k[\phi - \varphi]}$$

where $\phi = \langle \varphi \rangle$.

$$\Delta S_k[\phi] = \frac{1}{2} \int \phi R_k \phi$$

R_k has all the beautiful properties we ask, but in particular $R_{k=0} = 0$, so that $\Gamma_{k=0}$ is the usual effective action of QFT.

We know we reproduce Γ at $k = 0$,
but what exactly does it mean?

Perturbation theory

$Z = \int D\phi \, \mathrm{e}^{-S[\phi]}$

$\overline{\mathrm{MS}}$, μ

Scalar ϕ^4-theory:

$\beta_\lambda = \mu \frac{\partial \lambda}{\partial \mu} = \frac{3\lambda^2}{16\pi^2} - \frac{17\lambda^3}{768\pi^4}$

Functional RG

$k \frac{\partial}{\partial k} \Gamma_k[\phi] = \frac{\hbar}{2} \mathrm{Tr} \frac{k \frac{\partial}{\partial k} R_k}{\Gamma_k^{(2)}[\phi] + R_k}$

$\Gamma_k[\phi] = \sum_i g_i \mathcal{O}_i[\phi]$

Scalar ϕ^4-theory:

$\beta_\lambda = k \frac{\partial \lambda}{\partial k} = \frac{3\lambda^2}{16\pi^2 \left(1 + \frac{m^2}{k^2}\right)^3}$

Why the difference?

Answering this question in the most detailed possible way is fundamental to bridge a gap between the FRG "community" and any particle physicist.

We start by restoring $\hbar$ in the flow equation

$$k\frac{\partial}{\partial k}\Gamma_k[\varphi] = \frac{\hbar}{2}\mathrm{Tr}\frac{k\frac{\partial}{\partial k}R_k}{\Gamma_k^{(2)} + R_k}$$

Perform two expansions

$$\Gamma_k[\varphi] = S_{\mathrm{B}}[\varphi] + \sum_{L\geq 1}\hbar^L \Gamma_{L,\,k}[\varphi]$$

$$S_{\mathrm{B}}[\varphi] = S_{\mathrm{R}}[\varphi] + \sum_{L\geq 1}\hbar^L \delta S_L[\varphi]$$

Let us for the moment assume

$$\delta S_L = -\Gamma^{\mathrm{div}}_{L,\,k} \overset{!!!}{=} -\Gamma^{\mathrm{div}}_{L,\,k=0}$$

Each order in $\hbar$ has a flow equation

$$k\frac{\partial}{\partial k}\Gamma_{L,\,k}[\varphi] = \frac{1}{L!}\frac{\partial^L}{\partial \hbar^L}\left(k\frac{\partial}{\partial k}\Gamma_k[\varphi]\right)_{\hbar=0}$$

For example

$$k\frac{\partial}{\partial k}S_B[\varphi] = 0$$

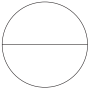
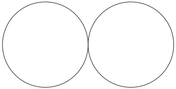
$$k\frac{\partial}{\partial k}\Gamma_{1,\,k}[\varphi] = \frac{1}{2}\mathrm{Tr}\frac{k\frac{\partial}{\partial k}R_k}{S_{\mathrm{B}}^{(2)}[\varphi]+R_k} = \frac{1}{2}\mathrm{Tr}\,k\frac{\partial}{\partial k}\log\left(S_{\mathrm{B}}^{(2)}[\varphi]+R_k\right)$$

We can integrate these flows separately but the process introduces UV-divergences!

$$\mathrm{Tr}\,k\frac{\partial}{\partial k} = k\frac{\partial}{\partial k}\,\mathrm{Tr}_{\mathrm{reg}}$$

After integration they reproduce exactly perturbation theory, if not for the presence of R_k

$$\Gamma_{1,\,k}[\varphi] = \frac{1}{2}\mathrm{Tr}_{\mathrm{reg}} \log\left(S_{\mathrm{B}}^{(2)}[\varphi] + R_k\right)$$

$$\Gamma_{2,\,k}[\varphi] = -\frac{1}{12}$$  $$+\frac{1}{8}$$ 

We can thus take a step backward and reconstruct the path-integral:

$$Z = \int [D\phi]_{\mathrm{reg}}\ \mathrm{e}^{-S[\phi]-\Delta S_k[\phi]}$$

$[D\phi]_{\mathrm{reg}}$ is the regularized measure we always implicitly assume in the FRG method.

Case study: φ^4 in $d = 4 - \epsilon$

$$S_B[\varphi] = \int \mathrm{d}^d x \left(\frac{1}{2} (\partial_\mu \varphi)^2 + \frac{m_{\mathrm{B}}^2}{2} \varphi^2 + \frac{\lambda_{\mathrm{B}}}{4!} \varphi^4 \right)$$

The theory is renormalized in the very standard way of $\overline{\mathrm{MS}}$-scheme, with the only difference that the propagator is modified by the IR-cutoff R_k. The requirement $\mu \frac{\partial}{\partial \mu} \lambda_{\mathrm{B}} = 0$ implies:

$$\beta_\lambda = -\epsilon \lambda + \frac{3\lambda^2}{16\pi^2} - \frac{17\lambda^3}{768\pi^4} + \dots$$

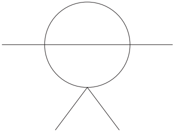
A consistency check:

Theorem

For a very general class of cutoffs R_k and in particular all those used in FRG, we have $\Gamma^{\mathrm{div}}_{L,\,k} = \Gamma^{\mathrm{div}}_{L,\,k=0} \Rightarrow \delta S_L$ is k independent.

Remark

The theorem above does not apply to single diagrams!
(And in fact subdivergences are dressed by k.)

divp 

$$= \frac{\lambda^3}{128\pi^4\varepsilon^2} + \frac{\lambda^3\left(4\log\left(\frac{\mu}{m}\right)-1\right)}{256\pi^4\varepsilon} - \frac{\lambda^3\left(\frac{1}{2}\log\left(1+\frac{k^2}{m^2}\right) - \frac{3k^4+2k^2m^2}{4(k^2+m^2)^2}\right)}{64\pi^4\varepsilon}$$

Since k plays no role in the renormalization, $\lambda(\mu)$ is the $\overline{\mathrm{MS}}$-coupling "at the scale μ". Let's call it $\lambda_{\overline{\mathrm{MS}}}(\mu)$.

The FRG coupling is defined implicitly by the expansion:

$$\Gamma_k[\varphi] = \int \mathrm{d}^4 x \frac{\lambda_{\mathrm{FRG}}(k)}{4!}\varphi^4 + \dots$$

From the finite parts, we have access to the perturbative info

$$\lambda_{\mathrm{FRG}}(k) = \lambda_{\overline{\mathrm{MS}}}(k) + \tfrac{3}{64\pi^2}\Bigg(2\log\left(\frac{k^2+m^2}{k^2}\right) + + \frac{-k^4 + 2k^2m^2 + 2m^4}{\left(k^2+m^2\right)^2}\Bigg)\lambda^2_{\overline{\mathrm{MS}}}(k)$$

and compute *using the 2-loop universal result*

$$\beta_{\mathrm{FRG}} = \frac{3\lambda^2_{\mathrm{FRG}}}{16\pi^2\left(1+\frac{m^2}{k^2}\right)^3} + \mathcal{O}\left(\lambda^3_{\mathrm{FRG}}\right)$$

The scheme change is summarized as:
$\left\{\lambda_{\overline{\mathrm{MS}}},\, m^2_{\overline{\mathrm{MS}}}\right\} \Longleftrightarrow \left\{\lambda_{\mathrm{FRG}},\, m^2_{\mathrm{FRG}},\, \lambda_{6,\mathrm{FRG}},\, \dots\right\}$
and does not belong to the “standard” class that preserves universality of the beta functions.

[D.F.Litim, J.M.Pawlowski: Phys.Rev.D66:025030,2002] for the loop expansion

[U.Ellwanger: Z.Phys. C76 (1997) 721-727] for the scheme change

[E.Manrique, M.Reuter: Phys.Rev.D79:025008,2009] for the scheme change

Questions?

The Papenbrock and Wetterich scheme.

1. Consider all operators that are generated at 1-loop.

$$\Gamma_k[\varphi] = \int \mathrm{d}^4x \Big(\frac{Z}{2} (\partial_\mu \varphi)^2 + \lambda_2 \varphi^2 + \lambda_4 \varphi^4 + \varphi^2 f_1(\Delta) \varphi^2 + \lambda_6 \varphi^6 + f_2(\Delta_1, \Delta_2, \Delta_3) \varphi_1^2 \varphi_2^2 \varphi_3^2 \Big)$$

2. Define the dimensionless renormalized couplings and form factors.

$$\lambda_2 = Z k^2 \tilde{\lambda}_2 \,, \qquad \varphi_R = \sqrt{Z} \varphi \,,$$
$$\lambda_4 = Z^2 \tilde{\lambda}_4 \,, \qquad f_1(q^2) = Z^2 \tilde{f}_1(q^2/k^2) \,,$$
$$\lambda_6 = Z^3 k^{-2} \tilde{\lambda}_6 \,, \qquad f_2(q_1^2, q_2^2, q_3^2) = Z^3 k^{-2} \tilde{f}_2(q_1^2/k^2, q_2^2/k^2, q_3^2/k^2)$$

3. Set all renormalized couplings and form factors *but* $\tilde{\lambda}_4$ at their FP as a function of λ_4 to the desired order.

Local couplings are easy to set:

$$G_q = (q^2 + R_k(q^2))^{-1}$$

$$k\frac{\partial}{\partial k}\tilde{\lambda}_2 = -2\tilde{\lambda}_2 - \frac{6\tilde{\lambda}_4}{k^2}\int_q G_q^2 k\frac{\partial}{\partial k}R_k(q^2) + \mathcal{O}(\lambda_4^3)$$

$$\tilde{\lambda}_{2*} = -\frac{3\tilde{\lambda}_4}{k^2}\int_q G_q^2 k\frac{\partial}{\partial k}R_k(q^2) + \mathcal{O}(\lambda_4^3)$$

Form factors are slightly more involved to deal with:

$$k\frac{\partial}{\partial k}\tilde{f}_1 - 2\eta\tilde{f}_1 - 2\tilde{f}_1'\frac{q^2}{k^2} = 72\tilde{\lambda}_4^2\int_Q \left(G_{Q+q} - G_Q\right) G_Q^2 k\frac{\partial}{\partial k}R_Q + \mathcal{O}\left(\lambda_4^3\right)$$

They are PDE that can be solved with the method of characteristics:

$$\begin{aligned}
\tilde{f}_{1*}(x) &= -\frac{\tilde{\lambda}_4^2}{2}\int_0^x \frac{\mathfrak{G}_1(y)}{y}\,\mathrm{d}y + \mathcal{O}\left(\lambda_4^3\right) \\
\mathfrak{G}_1\left(\frac{q^2}{k^2}\right) &\equiv 72\int_Q \left(G_{Q+q} - G_Q\right) G_Q^2 k\frac{\partial}{\partial k}R_Q
\end{aligned}$$

The form factors can be evaluated explicitly, but it proves convenient to manipulate them in such a way that the y integration cancels with the log k-derivative:

$$\begin{aligned} \tilde{f}_{1\,*}(x) &= -72\tilde{\lambda}_4^2\, I(q^2/k^2) \\ I(q^2/k^2) &= \frac{1}{2}\int_Q (G_{Q+q} - G_Q)G_Q \end{aligned}$$

This shows that the form factor at the FP takes the 1-loop form.

Remark.
$I(q^2/k^2)$ uses an implicit regularization that will play no role in the final result.

Repeat this for all other operators in Γ_k.

Inserting the FP values in the FRG beta-function of $\tilde{\lambda}_4$ we define the flow in the PW-scheme

$$\beta_{\tilde{\lambda}_4}(\eta_*(\tilde{\lambda}_4), \tilde{\lambda}_{2*}(\tilde{\lambda}_4), \tilde{\lambda}_4, \tilde{f}_{1*}(\tilde{\lambda}_4), \tilde{\lambda}_{6*}(\tilde{\lambda}_4), \tilde{f}_{2*}(\tilde{\lambda}_4)) \equiv \beta_{\rm PW}(\tilde{\lambda}_4)$$

$$\begin{aligned}
\beta_{\rm PW} \;\; = \;\; & 2\eta_*\tilde{\lambda}_4 + 72\tilde{\lambda}_4^2 \int_q G_q^3 \dot{R}_q - 432\tilde{\lambda}_4^2\tilde{\lambda}_{2*} \int_q G_q^4 \dot{R}_q \\
& +96\tilde{\lambda}_4 \int_q G_q^3 \dot{R}_q \; \tilde{f}_{1*}(q^2/k^2) - 15\frac{\tilde{\lambda}_{6*}}{k^2} \int_q G_q^2 \dot{R}_q \\
& -8\frac{\tilde{\lambda}_4^3}{k^2} \int_q G_q^2 \dot{R}_q \; \tilde{f}_{2*}(q^2/k^2) + \mathcal{O}\Big(\tilde{\lambda}_4^4\Big)
\end{aligned}$$

The "dot" is just a shorthand for $\partial/\partial \log k$.

Using the standard normalization

$$\tilde{\lambda}_4 \equiv \lambda_{\mathrm{PW}}/4!$$

the result displays universality

$$\begin{aligned} \eta_* &= \frac{1}{1536\pi^4}\lambda_{\mathrm{PW}}^2 + \mathcal{O}\left(\lambda_{\mathrm{PW}}^3\right) \\ \beta_{\mathrm{PW}} &= \frac{3}{16\pi^2}\lambda_{\mathrm{PW}}^2 - \frac{17}{768\pi^4}\lambda_{\mathrm{PW}}^3 + \mathcal{O}\left(\lambda_{\mathrm{PW}}^4\right) \end{aligned}$$

But do not get fooled, this is not the $\overline{\mathrm{MS}}$ coupling!

$$\lambda_{\mathrm{PW}}(k) = \lambda_{\overline{\mathrm{MS}}}(k) + \frac{\log 8 - 3\gamma}{32\pi^2}\lambda_{\overline{\mathrm{MS}}}^2(k)$$

[T.Papenbrock, C.Wetterich: Z.Phys. C65 (1995) 519-535]

Conclusions.

The questions we answered:

- What scheme is FRG and how does it relate to the others?
- How one-coupling beta functions and infinite-couplings beta functions relate?
- Does FRG violate universality?

The question we did not answer yet:

- Why the FRG method works so well?

An interesting attempt.

A truncation that encodes the effects of any desired operator $\Delta\Gamma_k[\varphi]$ and is at the same time 2-loops "universal" would be

$$\Gamma_k[\varphi] \;=\; \int \mathrm{d}^4x \Big(\frac{Z_*}{2} (\partial_\mu \varphi)^2 + \lambda_{2*}\varphi^2 + \lambda_{4*}\varphi^4 + \varphi^2 f_{1*}(\Delta)\varphi^2 + \lambda_{6*}\varphi^6 + f_{2*}(\Delta_1, \Delta_2, \Delta_3)\varphi_1^2\varphi_2^2\varphi_3^2 \Big) + \Delta\Gamma_k[\varphi]$$

Thank you for your attention.