Fermions in gravity with local spin-base invariance

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Motivation

aim at a theory of quantized gravity and quantized matter

need to know the fundamental degrees of freedom

guidance from classical GR is weak

- metric: $g_{\mu\nu}$
- vierbein: $e_{\mu}{}^{a}$
- vierbein and spin connection: $e_{\mu}{}^{a}$, $\omega_{\mu}{}^{a}{}_{b}$
- more complicated ones

  [Plebanski '77; Capovilla, Jacobson and Dell '89, '91; Ashtekar and Lewandowski '04; Krasnov '11]

all lead to the same classical equations of motion quantized theories are expected to differ, e.g.: $g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}$

leads to a non-trivial Jacobian between $\mathcal{D}g$ and $\mathcal{D}e$

explicit examples in FRG context:

- vierbein and spin connection [Daum and Reuter '12]
- vierbein [Harst and Reuter '12]
- chiral gravity [Harst, PhD thesis '13]

need criteria beyond pure mathematical consistency

fermions occur in our world
$\Rightarrow$ need vierbeins or even more fundamental objects

typically one introduces vierbeins ${e_\mu}^a$

$$g_{\mu\nu} = {e_\mu}^a {e_\nu}^b \eta_{ab}$$

the covariant derivative ${\nabla^{(e)}}_\mu$ of spinors ψ then reads

$${\nabla^{(e)}}_\mu \psi = \partial_\mu \psi + \frac{1}{8} {\omega_\mu}^{ab} [{\gamma^{(\mathrm{f})}}_a, {\gamma^{(\mathrm{f})}}_b] \psi$$

with spin connection ${\omega_\mu}^a{}_b$ and *flat* Dirac matrices ${\gamma^{(\mathrm{f})}}_a$

$${\nabla^{(e)}}_\mu {e_\nu}^a \equiv \partial_\mu {e_\nu}^a - \Gamma^\kappa_{\mu\nu} {e_\kappa}^a + {\omega_\mu}^a{}_b {e_\nu}^b \stackrel{!}{=} 0$$

$$\{{\gamma^{(\mathrm{f})}}_a, {\gamma^{(\mathrm{f})}}_b\} = 2\eta_{ab} \mathrm{I}$$

Issues:

- relevant objects for fermions are the Dirac matrices γ_μ

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathrm{I}$$

  $\Rightarrow$ more solutions than $\gamma_\mu = e_\mu{}^a \gamma^{(\mathrm{f})}{}_a$

- not the best choice of *basis* for some calculations [Finster, Smoller and Yau '99; Casals, Dolan, Nolan, Ottewill and Winstanley '13]
- special inertial coframe $e_\mu{}^a$ has to be introduced
- SO(3,1) symmetry for $e_\mu{}^a$ vs. SL(4,$\mathbb{C}$) symmetry for γ_μ

need criteria beyond pure mathematical consistency

fermions occur in our world

$\Rightarrow$ need vierbeins or even more fundamental objects... right?

reexamine the *necessity* of vierbeins in the presence of fermions

try to start from first principles

Spin-base invariance

Dirac structure:

- Clifford algebra (irreducible representation):
  $$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathrm{I}, \quad \gamma_\mu \in \mathbb{C}^{4\times 4}, \quad d_\gamma = d = 4$$

- spin-base invariance: $\mathcal{S} \in \mathrm{SL}(4, \mathbb{C})$
  $$\gamma_\mu \to \mathcal{S}\gamma_\mu\mathcal{S}^{-1}, \quad \psi \to \mathcal{S}\psi, \quad \bar{\psi} \to \bar{\psi}\mathcal{S}^{-1}$$

- Dirac conjugation with spin metric h:
  $$\bar{\psi} = \psi^\dagger h, \quad |\det h| = 1$$

[Finster '98, Weldon '01]

covariant derivative:

- linearity:
  $$\nabla_\mu(\psi_1 + \psi_2) = \nabla_\mu\psi_1 + \nabla_\mu\psi_2$$
- product rule:
  $$\nabla_\mu\psi\bar{\psi} = (\nabla_\mu\psi)\bar{\psi} + \psi(\nabla_\mu\bar{\psi})$$
- spin-base covariance:
  $$\nabla_\mu\bar{\psi} = \overline{\nabla_\mu\psi}, \quad \nabla_\mu\psi^\dagger = (\nabla_\mu\psi)^\dagger$$
- coordinate covariance:
  $$\nabla_\mu(\bar{\psi}\gamma^\nu\psi) = D_\mu(\bar{\psi}\gamma^\nu\psi) \equiv \partial_\mu(\bar{\psi}\gamma^\nu\psi) + \Gamma^\nu_{\mu\kappa}(\bar{\psi}\gamma^\kappa\psi),$$
  $$\Gamma^\nu_{\mu\kappa} = \left\{ {\nu \atop \mu\kappa} \right\} + K^\nu{}_{\mu\kappa}, \quad \left\{ {\nu \atop \mu\kappa} \right\} = \frac{1}{2}g^{\nu\rho}(\partial_\mu g_{\kappa\rho} + \partial_\kappa g_{\mu\rho} - \partial_\rho g_{\mu\kappa})$$

reality of action:

- mass term:
  $(\bar{\psi}\psi)^* = \bar{\psi}\psi$
- kinetic term:
  $\int \mathrm{d}^d x \sqrt{-g}\, (\bar{\psi}\nabla\!\!\!/\,\psi)^* = \int \mathrm{d}^d x \sqrt{-g}\, \bar{\psi}\nabla\!\!\!/\,\psi, \quad \nabla\!\!\!/\,\psi = \gamma^\mu \nabla_\mu \psi$

Spin metric and spin connection

construct spin metric h and spin covariant derivative ∇_μ

spin metric is implicitly determined through the γ^μ

Spin metric

$$\gamma_\mu^\dagger = -h\gamma_\mu h^{-1}, \quad h^\dagger = -h, \quad |\det h| = 1$$

spin covariant derivative:

$$\nabla_\mu \psi = \partial_\mu \psi + \hat{\Gamma}_\mu \psi + \Delta\Gamma_\mu \psi \qquad \Big| \; D_\mu v^\alpha = \partial_\mu v^\alpha + \left\{ {\alpha \atop \mu\beta} \right\} v^\beta + K^\alpha{}_{\mu\beta} v^\beta$$

canonical part of the spin connection $\hat{\Gamma}_\mu$

$$D_{(\mathrm{LC})\mu} \gamma^\nu = \partial_\mu \gamma^\nu + \left\{ {\nu \atop \mu\kappa} \right\} \gamma^\kappa = -[\hat{\Gamma}_\mu, \gamma^\nu], \quad \mathrm{tr}\, \hat{\Gamma}_\mu = 0$$

$$\hat{\Gamma}_\mu = p_\mu \gamma_* + v_\mu{}^\alpha \gamma_\alpha + a_\mu{}^\alpha \gamma_* \gamma_\alpha + t_\mu{}^{\alpha\beta} [\gamma_\alpha, \gamma_\beta],$$

$$\gamma_* = -\frac{\mathrm{i}}{4!} \sqrt{-g}\, \varepsilon_{\mu_1 \dots \mu_4} \gamma^{\mu_1} \dots \gamma^{\mu_4}$$

$$p_\mu = \frac{1}{32} \mathrm{tr}(\gamma_* \gamma_\alpha \partial_\mu \gamma^\alpha), \quad v_\mu{}^\alpha = \frac{1}{48} \mathrm{tr}([\gamma^\alpha, \gamma_\beta] \partial_\mu \gamma^\beta),$$

$$a_\mu{}^\alpha = \frac{1}{8} \mathrm{tr}(\gamma_* \partial_\mu \gamma^\alpha), \qquad t_{\mu\alpha}{}^\beta = -\frac{1}{32} \mathrm{tr}(\gamma_\alpha \partial_\mu \gamma^\beta) - \frac{1}{8} \left\{ {\beta \atop \mu\alpha} \right\}$$

spin torsion $\Delta\Gamma_\mu$:

$$0 = [\Delta\Gamma_\mu, \gamma^\mu], \quad \Delta\Gamma_\mu = -h^{-1}\Delta\Gamma_\mu^\dagger h$$

spin torsion carries 45 real parameters, but only 11 remain within the Dirac operator $\nabla\!\!\!/$:

$$\bar{\psi}\gamma^\mu\Delta\Gamma_\mu\psi = \mathscr{M}\bar{\psi}\psi - \mathscr{A}_\mu\bar{\psi}\mathrm{i}\gamma_*\gamma^\mu\psi - \mathscr{F}_{\mu\nu}\bar{\psi}\tfrac{\mathrm{i}}{4}[\gamma^\mu, \gamma^\nu]\psi$$

$\mathscr{M}$: mass/scalar field

$\mathscr{A}_\mu$: axial vector field

$\mathscr{F}_{\mu\nu}$: anti-symmetric tensor field to recover vierbein formalism, set ${\gamma^{(e)}}_\mu = {e_\mu}^a {\gamma^{(\mathrm{f})}}_a$ and find

$$\hat{\Gamma}_\mu + \Delta\Gamma_\mu = \frac{1}{8}{\omega_\mu}^{ab}[{\gamma^{(\mathrm{f})}}_a, {\gamma^{(\mathrm{f})}}_b]$$

$$\Delta\Gamma_\mu = \frac{1}{8}{K^\alpha}_\mu{}^\beta[{\gamma^{(e)}}_\alpha, {\gamma^{(e)}}_\beta]$$

the vierbein formalism gives an additional constraint on spacetime torsion: $\quad {K^\alpha}_{\alpha\mu} = 0$

or it violates: $\quad \int \mathrm{d}^d x \sqrt{-g}\ (\bar{\psi}\nabla\!\!\!/\,\psi)^* = \int \mathrm{d}^d x \sqrt{-g}\ \bar{\psi}\nabla\!\!\!/\,\psi$

summary part I

- impose full nontrivial symmetry of Clifford algebra
  $\Rightarrow$ spinbase transformations: SL(4, $\mathbb{C}$)
- impose *natural* conditions on ∇_μ
- spin metric h and canonical part of the spin connection $\hat{\Gamma}_\mu$ are determined through γ_μ
- vierbein formalism can be recovered (in spacetimes without torsion)

$\Rightarrow$ seems to suggest γ_μ as fundamental DoF

Questions?

Constructing an action

neglect torsion and spin torsion in the following (inclusion is not difficult, cf. [arXiv:1310.2509 [hep-th]]): $K^{\nu}{}_{\mu\kappa} = 0, \quad \Delta\Gamma_{\mu} = 0$

spinbase invariance is similar to gauge symmetry

$\Rightarrow$ define spin curvature (field strength) $\Phi_{\mu\nu}$

$$[\nabla_\mu, \nabla_\nu]\psi = \Phi_{\mu\nu}\psi$$

$\Rightarrow$ construct action S_Φ to lowest order in $\Phi_{\mu\nu}$

$$S_\Phi = \frac{1}{16\pi G}\int \mathrm{d}^d x \sqrt{-g}\, \frac{1}{4}\,\mathrm{tr}(\gamma^\mu \Phi_{\mu\nu}\gamma^\nu)$$

explicit calculation of $\Phi_{\mu\nu}$ gives

$$\begin{aligned}\Phi_{\mu\nu} &= \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu + [\Gamma_\mu, \Gamma_\nu] \\ &= \frac{1}{8} R_{\mu\nu\lambda\rho}[\gamma^\lambda, \gamma^\rho]\end{aligned}$$

the action then reads

$$\begin{aligned}S_\Phi &= \frac{1}{16\pi G} \int \mathrm{d}^d x \sqrt{-g}\ \frac{1}{8} R_{\mu\nu\rho\lambda} \frac{1}{4} \mathrm{tr}(\gamma^\mu [\gamma^\rho, \gamma^\lambda] \gamma^\nu) \\ &= \frac{1}{32\pi G} \int \mathrm{d}^d x \sqrt{-g}\ R\end{aligned}$$

$\Rightarrow$ S_Φ is equivalent to the Einstein-Hilbert action

Path integral

can construct action S if γ^μ are known

naive way: $\int \mathcal{D}\gamma \, \mathrm{e}^{\mathrm{i}S}$ (and fermions, gauge fields, ...)

but the Clifford algebra $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathrm{I}$ prohibits arbitrary variations of γ_μ

$\Rightarrow$ determine degrees of freedom from Clifford algebra

Weldon theorem [Weldon '01]

$$\delta\gamma^\mu = \frac{1}{2}(\delta g^{\mu\nu})\gamma_\nu + [\delta\mathcal{S}_\gamma, \gamma^\mu], \ \operatorname{tr}\delta\mathcal{S}_\gamma = 0$$

$\delta g_{\mu\nu}$: metric fluctuations

$\delta\mathcal{S}_\gamma$: spin-base fluctuations, $\mathrm{SL}(4,\mathbb{C})_\gamma$

want to show that the $\mathcal{S}_\gamma$ part factors out

ingredients for proof:

- action as a functional of the metric $g_{\mu\nu}$, the fermions ψ, $\bar{\psi}$ and the spin-base $\mathcal{S}_\gamma$: $S = S[\psi, \bar{\psi}, g; \mathcal{S}_\gamma]$
- spin-base invariance of S:
  $$S[\psi, \bar{\psi}, g; \mathcal{S}_\gamma] \to S[\mathcal{S}\psi, \bar{\psi}\mathcal{S}^{-1}, g; \mathcal{S}'_\gamma] \equiv S[\psi, \bar{\psi}, g; \mathcal{S}_\gamma]$$
- spin-base invariance of the measure $\mathcal{D}\psi\mathcal{D}\bar{\psi}$:
  $$\mathcal{D}\psi\mathcal{D}\bar{\psi} \to \mathcal{D}(\mathcal{S}\psi)\mathcal{D}(\bar{\psi}\mathcal{S}^{-1}) = \mathcal{D}\psi\mathcal{D}\bar{\psi}$$

study expectation value of an operator $\hat{O}(\psi, \bar{\psi}, g; \mathcal{S}_\gamma)$, which is scalar under spin-base transformations

consider integration over fermions and metric

$$\left\langle \hat{O}(\psi, \bar{\psi}, g; \mathcal{S}_\gamma) \right\rangle = \int \mathcal{D}g \mathcal{D}\psi \mathcal{D}\bar{\psi}\, \hat{O}(\psi, \bar{\psi}, g; \mathcal{S}_\gamma) \mathrm{e}^{\mathrm{i}S[\psi, \bar{\psi}, g; \mathcal{S}_\gamma]}$$

with spin-base invariance we find

$$\left\langle \hat{O}(\psi, \bar{\psi}, g; \mathcal{S}_\gamma) \right\rangle \equiv \left\langle \hat{O}(\psi, \bar{\psi}, g; \mathcal{S}'_\gamma) \right\rangle$$

$\Rightarrow$ integration over $\mathrm{SL}(4, \mathbb{C})_\gamma$ is trivial
$\Rightarrow$ in practice: fix spinbasis for purely metric-based quantization

(metric-based quantization also possible for Kähler fermions [Dona and Percacci '12])

Towards practical FRG computations

concrete quantum gravity calculations use propagators (or $\Gamma_k^{(2)}$)

need response of γ_μ up to second order in metric variations $\delta g_{\mu\nu}$

introduce fiducial background metric $\bar{g}$

$$\gamma_\mu(\bar{g}+\delta g) \simeq \bar{\gamma}_\mu + \left.\frac{\partial \gamma_\mu(g)}{\partial g_{\rho\lambda}}\right|_{g=\bar{g}} \delta g_{\rho\lambda} + \frac{1}{2} \left.\frac{\partial^2 \gamma_\mu(g)}{\partial g_{\alpha\beta} \partial g_{\rho\lambda}}\right|_{g=\bar{g}} \delta g_{\alpha\beta} \delta g_{\rho\lambda}$$

using the Weldon theorem

$$\delta\gamma_\mu = \frac{1}{2}\delta g_{\mu\nu}\gamma^\nu + [\delta\mathcal{S}_\gamma, \gamma_\mu]$$

and smoothness

$$\delta\mathcal{S}_\gamma = G^{\rho\lambda}\delta g_{\rho\lambda}$$

we get for the first derivative

$$\frac{\partial\gamma_\mu(g)}{\partial g_{\rho\lambda}} = \frac{1}{2}\delta^{\rho\lambda}_{\mu\nu}\gamma^\nu(g) + [G^{\rho\lambda}(g), \gamma_\mu(g)]$$

by choosing part of the spin base we can set: $G^{\rho\lambda}(g=\bar{g})=0$

for the second derivative we find (at the background $g=\bar{g}$)

$$\begin{aligned}\left.\frac{\partial^2\gamma_\mu(g)}{\partial g_{\alpha\beta}\partial g_{\rho\lambda}}\right|_{g=\bar{g}} &= -\frac{1}{4}\delta^{\rho\lambda}_{\mu\kappa}\bar{g}^{\kappa\sigma}\delta^{\alpha\beta}_{\sigma\nu}\bar{\gamma}^{\nu}+\left[\left.\frac{\partial G^{\rho\lambda}}{\partial g_{\alpha\beta}}\right|_{g=\bar{g}},\bar{\gamma}_\mu\right]\\ &= -\omega^{\rho\lambda\alpha\beta}{}_{\mu\nu}\bar{\gamma}^{\nu}+[G^{\alpha\beta\rho\lambda},\bar{\gamma}_\mu]\end{aligned}$$

second derivative must be symmetric under $(\alpha\beta) \leftrightarrow (\rho\lambda)$

$$\left.\frac{\partial^2 \gamma_\mu(g)}{\partial g_{\alpha\beta} \partial g_{\rho\lambda}}\right|_{g=\bar{g}} = -\omega^{\alpha\beta\rho\lambda}{}_{(\mu\nu)}\bar{\gamma}^\nu + \left[G^{\alpha\beta\rho\lambda} - \frac{1}{8}\omega^{\alpha\beta\rho\lambda}{}_{[\kappa\sigma]}[\bar{\gamma}^\kappa, \bar{\gamma}^\sigma], \bar{\gamma}_\mu \right]$$

by fixing even more of the spin base we can set:

$$G^{\alpha\beta\rho\lambda} = \frac{1}{8}\omega^{\alpha\beta\rho\lambda}{}_{[\kappa\sigma]}[\bar{\gamma}^\kappa, \bar{\gamma}^\sigma]$$

the *simplest* expansion for γ_μ then reads

$$\gamma_\mu(\bar{g}+\delta g) \simeq \bar{\gamma}_\mu + \frac{1}{2}\delta^{\rho\lambda}_{\mu\nu}\bar{\gamma}^\nu \delta g_{\rho\lambda} - \frac{1}{2}\omega^{\alpha\beta\rho\lambda}{}_{(\mu\nu)}\bar{\gamma}^\nu \delta g_{\alpha\beta}\delta g_{\rho\lambda}$$

it follows

$$h(g) = \bar{h} + \mathcal{O}(\delta g^3), \quad \gamma_*(g) = \bar{\gamma}_* + \mathcal{O}(\delta g^3)$$

equivalent to Lorentz symmetric gauge [Woodard '84]
used in FRG computations [Eichhorn, Gies '11]

summary part II

- metric-based quantization with fermions in principle possible
- need choice for spinbase, but possible by *hand* (integration of spin-base fluctuations not necessary)
- *simplest* choice similar to the commonly used Lorentz symmetric gauge within vierbein formalism

Thank you for your attention!