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# Brans-Dicke theory in the local potential approximation

Based on (hep-th:1311.1081) with **Dario Benedetti**

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# Motivations

- Recently $f(R)$ theories are gaining renovated interest for their applications in quantum gravity and cosmology.

- There is a strong evidence that gravity can be non-perturbatively renormalizable at a non-Gaussian fixed point, i.e. the **asymptotically safe scenario**.

- Solve the non perturbative RG flow equation for an $f(R)$, $R \in (-\infty, \infty)$, is not an easy task. Is it easier using equivalent theories?

- We can start with a metric $f(R)$ action

$$S[g_{\mu\nu}] = \int d^d x \sqrt{g}\, f(R)\,, \tag{1}$$

and write an equivalent theory in the Jordan frame, where

$$\phi = -\frac{d}{dR} f(R) \qquad f''(R) = -\frac{d}{dR}\phi(R) \neq 0\,, \tag{2}$$

$$V(\phi) = R(\phi)\,\phi + f(R(\phi))\,, \tag{3}$$

obtaining a scalar-tensor theory

$$S_{ST}[g_{\mu\nu}, \phi] = \int d^d x \sqrt{g}\, \{V(\phi) - \phi\, R\}\ . \tag{4}$$

that is a **Brans-Dicke** theory with $\omega = 0$ and generic potential $V(\phi)$.

- The scalar-tensor theory is easier to quantize (projectable of flat spacetime, no heat kernel issues, etc.). Are the theories equivalent at quantum level?

- We can search for scalar invariant solutions $V^*(\phi)$ of the RG equation in a local potential approximation, $\phi \in (-\infty, \infty)$, and invert back to global fixed point solution $f^*(R)$, where now $R \in (-\infty, \infty)$.

- We will quantize the most generic scalar-tensor theory

$$S_{ST}[g_{\mu\nu}, \phi] = \int d^d x \sqrt{g} \left\{ V(\phi) - \phi\, R + \frac{\omega}{\phi}\, \partial_\mu \phi\, \partial^\mu \phi \right\} . \tag{5}$$

# Quantization procedure

- We consider as bare action the Brans-Dicke action for generic potential $V(\phi)$ which reads

$$S[g,\phi] = \int d^d x \sqrt{g} \left\{ V(\phi) - \phi\, R + \frac{\omega}{\phi}\, \partial_\mu \phi\, \partial^\mu \phi \right\} . \tag{6}$$

- We will employ the background field method with vanishing background ghost fields, i.e.

$$g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu} \qquad \phi \to \phi + \varphi \,, \tag{7}$$

so that the path integral is formally defined as

$$Z[J,\beta_{\mu\nu};g,\phi] = \int \mathcal{D}[h]\mathcal{D}[\varphi]\, e^{-S[g+h,\phi+\varphi]-S_{gf}[h,\varphi;g,\phi]-S_{gh}[C,\bar{C},h,\varphi;g,\phi]+S_{sr}} \,, \tag{8}$$

being $S_{gf}$ and $S_{gh}$ respectively the gauge fixing and ghost actions and $S_{sr}$ the sources term.

# Gauge fixing: Feynman gauge

- The gauge fixing action reads

$$S_{gf} = \frac{1}{2\,a} \int d^d x \sqrt{g}\, \mathcal{F}_\mu G^{\mu\nu} \mathcal{F}_\nu \,, \tag{9}$$

being $\mathcal{F}_\mu$ a gauge-fixing constraint and $G^{\mu\nu}$ a non-degenerate operator.

We will employ two different gauges, namely a **Feynman** and a **Landau** gauge.

- The **Feynman** gauge $(a = 1)$ is obtained for a constraint operator

$$\mathcal{F}_\mu^{(F)} = \nabla^\nu \left( h_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} h \right) - \frac{1}{\phi} \nabla_\mu \varphi \,, \qquad G^{(F)\,\mu\nu} = \phi\, g^{\mu\nu} \,. \tag{10}$$

- The associated ghost action reads

$$S_{gh}[C, \bar{C}; g, \phi] = \int d^d x \sqrt{g} \left\{ \bar{C}^\mu \left( \nabla^2 + \frac{R}{d} \right) C_\mu \right\} . \tag{11}$$

- This gauge choices introduces a kinetic term for $\phi$ in the second variation of the action.

- A Landau gauge choice ($a = 0$) is performed by means of

$$\mathcal{F}_\mu^{(L)} = \nabla^\nu \left( h_{\mu\nu} - \frac{1}{d}\, g_{\mu\nu}\, h \right) , \qquad G^{(L)\,\mu\nu} = g^{\mu\nu} . \tag{12}$$

- In this case minimal operators are simplified by using a TT-decomposition, which introduces a Jacobian

$$S_{\mathrm{aux-gr}} = \int d^d x \sqrt{g} \left\{ 2\, \bar{\chi}^{T\,\mu} \left( \nabla^2 + \frac{R}{d} \right) \chi_\mu^T + \left( \frac{d-1}{d} \right) \bar{\chi} \left( \nabla^2 + \frac{R}{d-1} \right) \nabla^2 \chi \right\} + (\chi \to \zeta) , \tag{13}$$

$$S_{\mathrm{aux-gh}} = \int d^d x \sqrt{g}\, \eta\, \nabla^2\, \eta , \tag{14}$$

- Using the Landau gauge no kinetic operator for $\phi$ is introduced in the second variation.

# ERG equation

- The exact RG equation in its general form reads

$$\partial_t \Gamma_k[\psi] = \frac{1}{2}\mathrm{STr}\left[\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1} \partial_t \mathcal{R}_k\right] , \qquad (15)$$

being $\psi$ a generic superfield, $t \equiv \log(k/k_0)$.

- For the cutoff function $\mathcal{R}_k$ we choose a cutoff function which implements the substitution, $-\nabla^2 \to P_k \equiv -\nabla^2 + k^2 r(-\nabla^2/k^2)$, that is $\mathcal{R}_k = \Gamma_k^{(2)}|_{-\nabla^2 \to P_k} - \Gamma_k^{(2)}$, where $r(z)$ is the "optimized" cutoff function

$$r(z) = (1-z)\Theta(1-z) , \qquad (16)$$

- We will employ the common truncation for the effective action

$$\Gamma_k[h,\varphi; g,\phi] = \bar{\Gamma}_k[g+h, \phi+\varphi] + S_{\mathrm{gf}}[h,\varphi; g,\phi] + S_{\mathrm{gh}}[\bar{C}, C, h, \varphi; g, \phi] , \qquad (17)$$

so that

$$\bar{\Gamma}_k[g,\phi] = \int d^d x \sqrt{g}\left\{ V_k(\phi) - \phi\, R + \frac{\omega}{\phi}\,\partial_\mu\phi\,\partial^\mu\phi \right\} . \qquad (18)$$

- The ERG equation for the dimensionless potential is obtained working in dimensionless quantities

$$\tilde{\phi} = \phi\, k^{2-d}\,, \quad \tilde{V}(\tilde{\phi}) = V(k^{d-2}\tilde{\phi})\, k^{-d}\,, \tag{19}$$

so that we have

$$\partial_t\, \tilde{V}_k(\tilde{\phi}) = \mathcal{T}_{\rm tree} + \mathcal{T}_{\rm quant}\,, \tag{20}$$

where

$$\mathcal{T}_{\rm tree} = -d\, \tilde{V}(\tilde{\phi}) + (d-2)\, \tilde{\phi}\, \tilde{V}'(\tilde{\phi})\,, \tag{21}$$

$$\mathcal{T}_{\rm quant} = \mathcal{A}(d) \left\{ \mathcal{B} + \frac{\mathcal{C}(d)\, \tilde{\phi}}{\left(\tilde{\phi} - \tilde{V}(\tilde{\phi})\right)} + \frac{\widetilde{N}}{\widetilde{D}} \right\}\,, \tag{22}$$

where $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are constants and $\widetilde{N}$ and $\widetilde{D}$ are functions of $\tilde{\phi}$ and $\tilde{V}$ (they contain $\tilde{V}''(\tilde{\phi})$!) and their expression depends on the gauge choice.

- Fixed point solutions of the LPA equation satisfy

$$\partial_t \, \tilde{V}_k(\tilde{\phi}) = 0 \,. \tag{23}$$

- **Strategy**: rewrite the equation in normal form

$$\tilde{V}_k''(\tilde{\phi}) = \frac{\mathcal{N}(\tilde{V}, \tilde{V}', \tilde{\phi})}{\mathcal{D}(\tilde{V}, \tilde{V}', \tilde{\phi})} \,, \tag{24}$$

1. study of asymptotic solutions at $\tilde{\phi} \to \infty$ ,
2. investigate fixed and movable singularities ,
3. integrate numerically from $\tilde{\phi} = 0$ (if possibile) .

# Questions?

- The large field regime of the differential equation can be studied using the method of dominant balance. We found in both gauges, for $\omega = 0$ and $d = 4$, a class of solutions of the type

$$\tilde{V}(\tilde{\phi} \to \pm\infty) \sim A\,\tilde{\phi}^2 \left(1 + \sum_{n>0} a_n(A)\,\tilde{\phi}^{-n}\right), \tag{25}$$

where the coefficients $a_n(A)$ are proportional to inverse powers of $A$, thus $A \neq 0$.

- Perturbing the ansatz (25) we found exponentially small subleading corrections, so that a more general solution reads

$$\tilde{V}(\tilde{\phi} \to \pm\infty) \sim \sum_{m=0}^{\infty} \left(B\,e^{Z_A(\tilde{\phi})}\right)^m \tilde{V}_{(m)}(\tilde{\phi}, A)\,, \tag{26}$$

where $V_{(0)}(\tilde{\phi}, A)$ is (25), and in the Feynman gauge $B = 1$ and $Z(\phi) = 0$ for $\tilde{\phi} \to +\infty$ (not in Landau).

# Asymptotic behavior

- For $A = 0$ the ansatz is

$$\tilde{V}(\tilde{\phi} \to \pm\infty) \sim A_1 \, \tilde{\phi} + \sum_{n=0}^{\infty} b_n(A_1) \, \phi^{-n} \tag{27}$$

which leads to

1. $\underline{A_1 = 0}$, with $b_n$ constants and exponential subleading corrections carrying one d.o.f. for $\phi \to -\infty$ in both gauges,
2. $\underline{A_1 = 1}$, with $b_n$ constants and exponential subleading corrections carrying one d.o.f. for $\phi \to \pm\infty$ in both gauges,
3. $\underline{A_1 = \frac{3}{2}}$, where $b_n$ are constants in the Feynman gauge. In the Landau gauge they bring a degree of freedom $B$, $b_n \equiv b_n(B)$, with $n > 0$, plus logarithmic corrections (that bring a new d.o.f. $C$).

- The ODE equation has just a fixed singularity at $\tilde{\phi} = 0$. Expanding $\tilde{V}''(\tilde{\phi})$ in a Laurent series

$$\tilde{V}''(\tilde{\phi}) = y_{-1}\frac{1}{\phi} + y_0 + y_1\tilde{\phi} + \mathcal{O}(\tilde{\phi}^2)\,. \tag{28}$$

- **Feynman** gauge: analyticity $(y_{-1} = 0)$ for: $\omega = -\frac{1}{2}$ or $\tilde{V}(0) = C_F(\omega)$ ,
- **Landau** gauge: analyticity $(y_{-1} = 0)$ for: $\omega = 0$ or $\tilde{V}(0) = C_L(\omega)$ ,

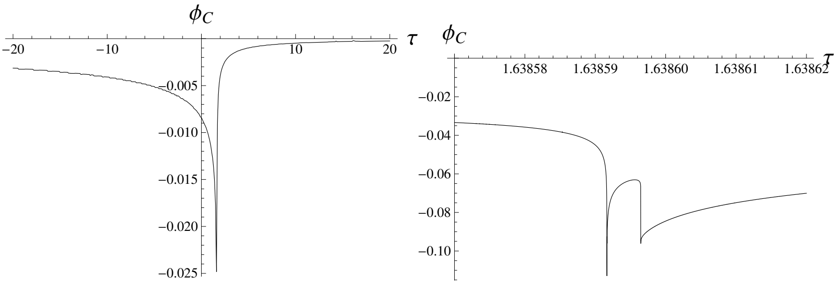
- We have then (for $\omega = 0$):
  1. two d.o.f. $(\tilde{V}(0), \tilde{V}'(0) = \tau)$ at the origin for the **Landau** gauge
  2. and just one d.o.f. $(\tilde{V}'(0) = \tau)$ at the origin in the **Feynman** case.

- Because of non linearity of the PDE we found two movable singularities reached at $\tilde{\phi}_c$, in both gauges, and they go like

$$\tilde{V}(\tilde{\phi}) = (\tilde{\phi} - \tilde{\phi}_c)^\gamma \left\{A + A_1\,(\tilde{\phi} - \tilde{\phi}_c) + \mathcal{O}((\tilde{\phi} - \tilde{\phi}_c)^2)\right\} + U(\tilde{\phi} - \tilde{\phi}_c)\,, \tag{29}$$

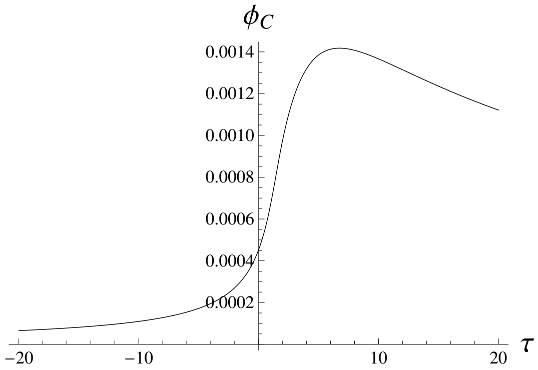
with $\gamma = \frac{3}{2}$ and $\gamma = -1$.

## Fixed point: **Feynman** $(\phi < 0, \omega = 0)$




- The two peaks correspond to a change in the singular behavior from $\gamma = \frac{3}{2}$ to $\gamma = -1$.

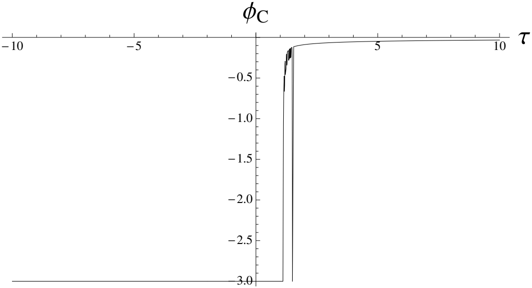
## Fixed point: **Feynman** $(\phi > 0, \omega = 0)$




- So far no fixed points has been found in the **Feynman** gauge.

# Fixed point: **Landau** ($\tilde{V}(0) > 0, \omega = 0$)

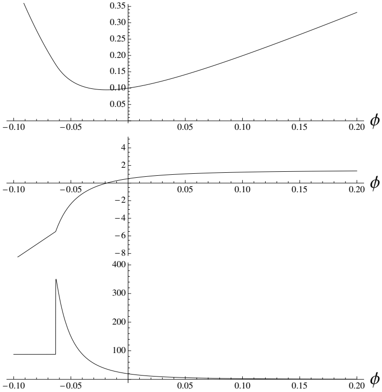
In the case of the Landau gauge we have an additional initial condition $\tilde{V}(0)$.
For $\tilde{V}(0) > 0$ and $\tilde{\phi} < 0$ we have




- There is a 2-parameter family of solutions with $\tilde{V}(\tilde{\phi} \sim -\infty) \sim A\,\tilde{\phi}^2$.
- The peak at $\tau = 1.5$ correspond to a linear potential $\tilde{V}(\phi) = A + \frac{2\,(d-1)}{d}\tilde{\phi}$ for which the second variation is singular, i.e. is not a solution.

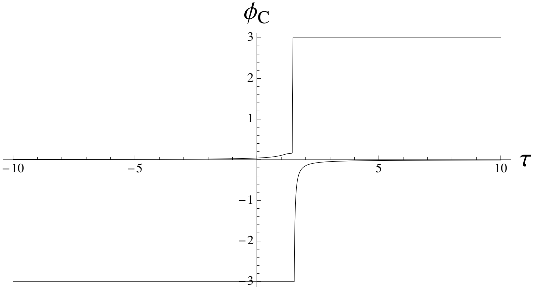
# Fixed point: **Landau** ($\tilde{V}(0) > 0, \omega = 0$)

- For $\tilde{\phi} > 0$ no singularities are encountered, asymptotic behaviors are:
  1. $\tilde{V}(\tilde{\phi} \sim -\infty) \sim A\,\tilde{\phi}^2$ for $\tau > 1.5$ ,
  2. $\tilde{V}(\tilde{\phi} \sim -\infty) \sim \frac{3}{2}\tilde{\phi}$ for $\tau < 1.5$ ,
- Solution, first and second derivatives for $\tilde{V}(0) = 0.1$, $\tau = 0.5$ :



# Fixed point: **Landau** ( $\tilde{V}(0) < 0, \omega = 0$)

- No global solutions have been found for $\tilde{V}(0) < 0$ :



# Conclusion and discussion

- We presented here an ERG flow equation for the Brans-Dicke theory with generic potential $V(\phi)$ and parameter $\omega$, employing a Landau and Feynman gauge.
- We focused on the case $\omega = 0$ in light of the classical equivalence with an $f(R)$ theory and we searched for fixed point solutions $\tilde{V}^*(0)$.
- We found no solutions in the Feynman gauge and a 2-dimensional continuos set of solutions in the Landau gauge.
- Quantum equivalence? Since $R = V'(\phi)$ if at $\phi_c$ we have $V'(\phi_c) = \infty$ then $\phi_c$ is mapped to $R \to \infty$. Vice versa, global solutions in Landau gauge with $V'(\infty) \sim \frac{3}{2}$ are mapped to $R = \frac{3}{2}$, thus are not global f(R) solutions.
- Those results have been obtained in a local potential approximation. A future work can take in account the running of the coupling $\omega \equiv \omega_k$ and the coupling $Z \equiv Z_k$ of the $\phi\, R$ operator.

Thanks for the attention.