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One-loop renormalization in a toy model of Hořava-Lifshitz gravity

Based on (hep-th:1311.6253) with **Dario Benedetti**

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Berlin, 03.02.2014

- General relativity is perturbatively non-renormalizable. String theory? Asymptotic safety scenario?
- Power-counting renormalizability can be reconciled with unitarity at the expenses of a scale anisotropy between space and time. Hořava-Lifshitz gravity.
- Not much is known about the behaviour Hořava-Lifshitz gravity in the UV. Difficulties coming from the large number of invariants and working on an anisotropic curved background.

- General relativity is power-counting non-renormalizabile in $d > 2$. $[G] = 2 - d$.

- Higher-derivative theories can cure the perturbative non-renormalizability, but they suffer from a lack of unitarity.

$$S[g_{\mu\nu}]_{hd} = -\frac{1}{16\,\pi\,G}\int d^dx\,\sqrt{g}\,\{R - \gamma\,R^2 - \beta\,R_{\mu\nu}\,R^{\mu\nu} - 2\Lambda\}\,, \qquad (1)$$

- The propagator of spin-2 modes contains a 'poltergeist' which spoils unitarity.

$$\Pi(p) = \frac{1}{p^2 - \beta\,G\,p^4}\,, \qquad p^2_{ghost} = (\beta\,G)^{-1}\,. \qquad (2)$$

- Power-counting renormalizability can be obtained by means of anisotropic scale dimensions for space and time coordinates, emulating what happens in condensed matter.

$$[\mathbf{x}] = -1, \qquad [t] = -z\,, \qquad\qquad [\partial_{\mathbf{x}}] = 1, \qquad [\partial_t] = z\,. \tag{3}$$

- As an example, for a Lifshitz scalar field theory we have

$$S[\phi] = \int dt\, d^d\mathbf{x} \left\{ -\phi(t,\mathbf{x})\,(\partial_t^2 - G\,\partial_{\mathbf{x}}^{2\,z})\,\phi(t,\mathbf{x}) \right\}\,. \tag{4}$$

- The propagator of the Lifshitz scalar field is

$$\frac{1}{\omega^2 - G(\mathbf{k}^2)^z}\,, \tag{5}$$

and a term $-c^2\,\partial_{\mathbf{x}}^2$ in the action acts now (for $z > 1$) as a relevant perturbation.

- Introducing a scale anisotropy we can gain perturbative-power counting and unitarity, but we lose Lorentz invariance.

  Invariance under diffeomorphisms is substituted by the invariance under a foliation-preserving diffeomorphisms, i.e. the reparameterization

$$\tilde{x}^i = \tilde{x}^i(x^j, t), \qquad \tilde{t} = \tilde{t}(t)\,. \tag{6}$$

- We have to build invariants under the new symmetry group.

  We employ the ADM decomposition, with g_{ij} the spatial metric, N_i the shift vector and N the lapse function.

- A natural spacetime topology in presence of a foliation $\mathcal{F}$ is

$$\mathcal{M} = \mathcal{N} \times \Sigma\,. \tag{7}$$

where the leaf Σ is a generic d-dimensional manifold. We will consider $\mathcal{N} = \mathbb{R}$.

- A kinetic action in $d+1$ dimensions reads

$$S_K[N, N^i, g_{ij}] = \frac{2}{\kappa^2} \int dt\, d^d\mathbf{x} \sqrt{g}\, N \left(K_{ij}K^{ij} - \lambda K^2\right) , \tag{8}$$

being

$$K_{ij} = \frac{1}{2N} \left(\partial_t\, g_{ij} - D_i N_j - D_j N_i\right) , \tag{9}$$

where D_i is the spatial covariant derivative.

- At this stage the dynamical critical exponent z enters only in the dimension of the integration measure

$$[dt\, d^d\mathbf{x}] = -d - z\,. \tag{10}$$

- As a consequence, Newton's constant κ^2 is a marginal parameter for $z = d$, since

$$[\kappa] = \frac{z-d}{2}\,. \tag{11}$$

- We can add a potential term by constructing spatial invariants

$$S[N, N^i, g_{ij}] = S_K[N, N^i, g_{ij}] + \frac{2}{\kappa^2} \int dt\, d^d\mathbf{x} \sqrt{g}\, N\; V(g_{ij})\,. \qquad (12)$$

- In 3+1 dimensions with $z = 3$ the potential $V(g_{ij})$ will contain marginal operators like

$$R^3\,, \qquad R^{ij}\, R_{jk}\, R^k{}_i\,, \qquad R\, R_{ij} R^{ij} \qquad D^2 R^2 \qquad D_k R_{ij}\, D^k R^{ij} \qquad \cdots \qquad (13)$$

plus additional relevant terms as

$$R^2\,, \qquad R_{ij}\, R^{ij}\,, \qquad D^2\, R \qquad D^i\, D^j\, R_{ij}\,, \qquad \cdots \qquad (14)$$

- The number of invariants is reduced if the system features detailed balance, that is, if the $d+1$-dimensional potential is related to a d-dimensional potential by means of a variational principle.

- The $2+1$-dimensional case is easier to study, since for $z=2$ the potential contains fewer invariants. Furthermore, the Ricci is the only independent component of the Riemann tensor study and there are no gravitons.

- The theory with detailed balance has no potential.

- The most general action (non-projectable and without detailed balance) in $2+1$ dimensions with $z=2$ is

$$S[N,N_i,g_{ij}] = \int dt\, d^2x N\sqrt{g}\left\{\frac{2}{\kappa^2}\left(\lambda\, K^2 - K_{ij}K^{ij} - 2\,\Lambda + c\,R + \gamma\, R^2\right) + c_1\, D^2\, R \right.$$
$$+ c_2\, a_i\, a^i + c_3\,(a_i\, a^i)^2 + c_4\, R\, a_i\, a^i + c_5\, a_i\, a^i\, D^j\, a_j$$
$$\left. + c_6\,(D^j\, a_j)^2 + c_7\,(D_i\, a_j)(D^i\, a^j)\right\}\,, \tag{15}$$

where a_i is the acceleration vector, $a_i = D_i \log N$.

We will consider two simplifications:

- **Projectable case**, that is, a constant lapse function over the leaf, $N \equiv N(t)$, so that

$$S[N, N_i, g_{ij}] = \frac{2}{\kappa^2} \int dt\, d^2x N \sqrt{g} \left\{ \lambda\, K^2 - K_{ij} K^{ij} - 2\,\Lambda + c\, R + \gamma\, R^2 \right\} . \quad (16)$$

- **Conformal reduction**, that is, a scalar toy model in which we integrate only the conformal degree of freedom of the spatial metric,

$$g_{ij} = e^{2\,\phi(x)}\, \tilde{g}_{ij} \,, \quad (17)$$

where $\tilde{g}_{ij}$ is considered as a constant background. We will chose $\Sigma = S^2$.

Questions?

- We will employ as usual the background field method:

$$g_{ij} \to g_{ij} + \epsilon\, h_{ij}\,; \quad N \to N + \epsilon\, n\,; \quad N_i \to N_i + \epsilon\, n_i\,, \tag{18}$$

  and set $N = 1$ and $N_j = 0$ for the background lapse and shift.

  The perturbative parameter ϵ will be set at a later stage.

- The metric fluctuation can be decomposed in a trace and traceless parts as

$$h_{ij} = \hat{h}_{ij} + \frac{1}{2}\, g_{ij}\, h\,, \qquad g^{ij}\hat{h}_{ij} = 0. \tag{19}$$

- On a sphere S^2 the traceless part $\hat{h}_{ij}$ contains just longitudinal components. In our toy model we neglect those contribution and integrate only the trace term.

Gauge-fixing and ghosts

- We employ as a gauge choices $n = 0$ and $n_i = 0$. The gauge-fixing action reads

$$S_{gf} = \frac{1}{2\,\alpha^2} \int dt \int d^2x \sqrt{g}\; n^2 + \frac{1}{2\,\beta^2} \int dt \int d^2x \sqrt{g}\; n_i\, n^i\,. \qquad (20)$$

Taking the limit $\alpha \to 0$ and $\beta \to 0$ leads to a complete decoupling of n and n_i in the second variation of the action.

- For the ghost sector, in order to avoid positivity problems of the Faddeev-Popov operator $\mathcal{M} = \partial_t$ we will employ the squared root of its determinant, $\sqrt{\det(-\mathcal{M}^2)}$, which corresponds to the ghost action

$$S_{gh} = \int dt\, N \int d^2x \sqrt{g} \left\{ \bar{c}\, \partial_t^2\, c + \bar{c}_i\, \partial_t^2\, c^i + b\, \partial_t^2\, b + b_i\, \partial_t^2\, b^i \right\}, \qquad (21)$$

where c_i and c Grassmannian complex fields and b_i and b real bosonic fields.

- We will focus on evaluating only the one-loop correction to the effective action,

$$\Gamma = S_{tot} + \hbar\, S^{1-loop} + \mathcal{O}(\hbar^2)\,, \tag{22}$$

being

$$S_{tot} = S + S_{gf} + S_{gh}\,, \tag{23}$$

and where

$$S^{1-loop} = \frac{1}{2}\,\mathrm{STr}\ln S^{(2)}_{tot}\,. \tag{24}$$

- The Hessian can be evaluated by the reduced second variation of the action, i.e.

$$\delta^2 S = \frac{1}{2\kappa^2}\int dt\, d^2x\,\sqrt{g}\,\left\{\left(\lambda - \frac{1}{2}\right)(\partial_t h)^2 + \gamma\, h\,(D^4 + 2\,R\,D^2 + R^2)\,h\right\}. \tag{25}$$

- The perturbative parameter is chosen so to normalize the kinetic operator, so that in our case

$$\epsilon = \frac{\kappa}{(\lambda - \frac{1}{2})^{\frac{1}{2}}}\,. \tag{26}$$

- We will employ heat-kernel techniques to evaluate the one-loop term, i.e.

$$S^{1-loop} = \frac{1}{2}\,\mathrm{Tr}\ln(\mathcal{S}^{(2)}) = -\frac{1}{2}\int_{\frac{1}{\Lambda^4}}^{\frac{1}{\mu^4}} \frac{ds}{s}\,\mathrm{Tr}\,\mathcal{H}(x,s;\mathcal{S}^{(2)}) = \\ -\frac{1}{2}\int_{\frac{1}{\Lambda^4}}^{\frac{1}{\mu^4}} \frac{ds}{s^2}\int dt\,d^2x\sqrt{\hat{g}}\left\{a_0 + s^{\frac{1}{2}}\,a_1 + s\,a_2 + \mathcal{O}(s^{\frac{3}{2}})\right\}, \tag{27}$$

where $a_2 = b_1\,K_{ij}K^{ij} + b_2\,K^2 + b_3\,R^2 + \cdots$, and μ is a renormalization scale.

- For the coefficients b_1 and b_2 we can use the result of Baggio, de Boer and Holsheimer (1112.6416) for an anisotropic differential operator $\mathcal{D}$ action on a scalar field, i.e.

$$\mathcal{D}_{bbh} = -\frac{1}{N\sqrt{g}}\,\partial_t\frac{1}{N}\sqrt{g}\,\partial_t + D^4\,, \tag{28}$$

whereas our operator reads instead

$$\mathcal{D} = -(\lambda - \frac{1}{2})\,\frac{1}{\sqrt{g}}\,\partial_t\sqrt{g}\,\partial_t + \gamma\,(D^2 + R)^2\,. \tag{29}$$

- As already mentioned, using the expression of ϵ in the background field decomposition the kinetic part normalizes, i.e.

$$\mathcal{D} = -\frac{1}{\sqrt{g}}\,\partial_t \sqrt{g}\,\partial_t + \frac{\gamma}{\lambda - \frac{1}{2}}\,(D^2 + R)^2\,. \tag{30}$$

- The normalization of the spatial part can be obtained by working with an auxiliary metric,

$$\hat{g}_{ij} = \left(\frac{\lambda - \frac{1}{2}}{\gamma}\right)^{\frac{1}{2}} g_{ij}\,, \tag{31}$$

so that we have

$$\hat{\mathcal{D}} = -\frac{1}{\sqrt{\hat{g}}}\,\partial_t \sqrt{\hat{g}}\,\partial_t + (\hat{D}^2 + \hat{R})^2\,. \tag{32}$$

- We can use the results of Baggio et.al. and obtain for b_1 and b_2

$$a_2 = -\frac{1}{256\,\pi}\left(\hat{K}_{ij}\,\hat{K}^{ij} - \frac{1}{2}\,\hat{K}^2\right) + \cdots\,. \tag{33}$$

Heat-kernel techniques

- For the coefficient b_3 we can use the well known results for higher-derivative operators in the isotropic case (see Gusynin, Nucl.Phys. B333).

  In particular, the heat kernel coefficient of R^2 for an operator $(D^2+X)^2$ vanishes in two dimensions, so that we have no R^2 term in our one-loop result.

- The one-loop correction is then equal to

$$\frac{1}{2}\,\hat{T}r\ln(\hat{\mathcal{D}}) = -\,\frac{1}{2}\int dt\, d^2x\sqrt{\hat{g}}\left\{(\Lambda^4-\mu^4)\,\frac{1}{16\,\pi}+(\Lambda^2-\mu^2)\,\frac{14}{48\,\pi^{3/2}}\hat{R}\right. \\ \left. +\ln\left(\frac{\Lambda}{\mu}\right)\,\frac{1}{16\,\pi}\left\{-\frac{1}{4}\,\hat{K}_{ij}\,\hat{K}^{ij}+\frac{1}{8}\,\hat{K}^2\right\}+\mathcal{O}\left(\frac{1}{\Lambda^2}\right)\right\}. \quad (34)$$

- We are interested only in the logarithmic divergence, which rewritten in terms of the physical metric reads

$$S^{1-loop}_{log} = \frac{1}{128\,\pi}\left(\frac{\lambda-\frac{1}{2}}{\gamma}\right)^{\frac{1}{2}}\ln\left(\frac{\Lambda}{\mu}\right)\int dt\, d^2x\sqrt{g}\,\left\{K_{ij}\,K^{ij}-\frac{1}{2}\,K^2\right\}. \quad (35)$$

β-functions

- We can obtained the β-functions by expressing the i-th bare coupling $g_{b,i}$ as $g_{b,i} = g_{R,i} + \delta g_i$, being $g_{R,i}$ a renormalized coupling δg_i a counterterm, so that

$$\begin{aligned} \frac{2}{\kappa_R^2} &= \frac{2}{\kappa^2} - \frac{1}{128\,\pi}\left(\frac{\lambda - \frac{1}{2}}{\gamma}\right)^{\frac{1}{2}} \ln\left(\frac{\Lambda}{\mu}\right), \\ \frac{2\,\lambda_R}{\kappa_R^2} &= \frac{2\,\lambda}{\kappa^2} - \frac{1}{256\,\pi}\left(\frac{\lambda - \frac{1}{2}}{\gamma}\right)^{\frac{1}{2}} \ln\left(\frac{\Lambda}{\mu}\right), \\ \frac{2\,\gamma_R}{\kappa_R^2} &= \frac{2\,\gamma}{\kappa^2}\,. \end{aligned} \tag{36}$$

- The renormalized couplings then read

$$\begin{aligned} \kappa_R^2 &= \kappa^2\left(1 + \frac{\kappa^2}{256\,\pi}\left(\frac{\lambda - \frac{1}{2}}{\gamma}\right)^{\frac{1}{2}} \ln\left(\frac{\Lambda}{\mu}\right)\right) + \mathcal{O}(\hbar^2)\,, \\ \lambda_R &= \lambda + \frac{1}{256\,\pi}\frac{\kappa^2}{\gamma^{1/2}}\left(\lambda - \frac{1}{2}\right)^{\frac{3}{2}} \ln\left(\frac{\Lambda}{\mu}\right) + \mathcal{O}(\hbar^2)\,, \\ \gamma_R &= \gamma\left(1 + \frac{\kappa^2}{256\,\pi}\left(\frac{\lambda - \frac{1}{2}}{\gamma}\right)^{\frac{1}{2}} \ln\left(\frac{\Lambda}{\mu}\right)\right) + \mathcal{O}(\hbar^2)\,. \end{aligned} \tag{37}$$

- As usual the β-functions are obtained stating $\partial_\mu\, g_b = 0$, so that

$$\begin{aligned} \beta_{\kappa^2} &= \mu\,\partial_\mu\,\kappa_R^2 = -\frac{\kappa^4}{256\,\pi}\left(\frac{\lambda-\frac{1}{2}}{\gamma}\right)^{\frac{1}{2}}, \\ \beta_\lambda &= \mu\,\partial_\mu\,\lambda_R = \frac{\left(\lambda-\frac{1}{2}\right)}{\kappa^2}\,\beta_{\kappa^2}\,, \\ \beta_\gamma &= \mu\,\partial_\mu\,\gamma_R = \frac{\gamma}{\kappa^2}\,\beta_{\kappa^2}\,. \end{aligned} \tag{38}$$

- Solving the system we find

$$\begin{aligned} k_R^2(\mu) &= \frac{256\,\pi}{b^{1/2}\,(\ln\frac{\mu}{\mu_0}+C)}\,, \\ \lambda_R(\mu) &= \frac{1}{2}+\frac{C_1}{\ln\frac{\mu}{\mu_0}+C}\,, \\ \gamma_R(\mu) &= \frac{C_2}{\ln\frac{\mu}{\mu_0}+C}\,. \end{aligned} \tag{39}$$

where C and $b = C_1/C_2$ are integration constants.

- Although the Newton's constant tends to zero in the UV, the interaction of the theory is defined by ϵ, whose running reads

$$\epsilon_R^2 = \frac{\kappa_R^2}{\lambda_R - \frac{1}{2}} = \frac{256\,\pi\, C_2^{1/2}}{C_1^{3/2}}\,. \tag{40}$$

Consequently, at one loop the theory is not asymptotically free.

- However, it is interesting to note that λ tends to $1/2$. For this value the kinetic action is invariant under anisotropic Weyl transformations

$$g_{ij} \to e^{2\,\phi(t,\mathbf{x})}\, g_{ij}\,, \qquad N \to e^{z\,\phi(t,\mathbf{x})}\, N\,, \qquad N_i \to e^{2\,\phi(t,\mathbf{x})}\, N_i\,. \tag{41}$$

We expect the running of λ to the conformal point to be a feature of the full model.

Conclusions and discussion

- Hořava-Lifshitz gravity in $d+1$ dimensions with $z=d$ features simultaneously power-counting renormalizability and unitarity.

- Because of the large number of couplings and complications coming from the anisotropic character of the background the UV has not been explored.

- We studied a simpler case, i.e. Hořava-Lifshitz gravity in 2+1 dimensions with $z=2$. We focused on a conformally reduced version of the projectable case.

- At one-loop the Newton's constant runs to zero in UV and λ to its conformal value $\lambda=\frac{1}{2}$. The interaction strength, however, is constant along the flow.

- What happens at two loops? What happens with gravitons? We expect λ to run to its conformal value also in the full model.

Thanks for the attention.