RG flows of Quantum Einstein Gravity on maximally symmetric spaces

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setup

- tool: $k\frac{\mathrm{d}\Gamma_k}{\mathrm{d}k} = \frac{1}{2}\,\mathrm{Tr}\left[\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1} k\frac{\mathrm{d}\mathcal{R}_k}{\mathrm{d}k}\right]$
- truncation: $\Gamma_k^{\mathrm{grav}}[g_{\mu\nu}] = \int \mathrm{d}^3x\,\sqrt{g}\,f_k(R)$
  [0705.1769, 0712.0445, 1204.3541, 1211.0955, ...]
- conformal reduction: $h_{\mu\nu} = \frac{1}{d}\bar{g}_{\mu\nu}\,\phi$
  [0801.3287, ...]
- maximally symmetric background: S^3 $(R > 0)$ and H^3 $(R < 0)$

type II regulator

operator: $\Box := -\bar{D}^2 + \mathbf{E}$ (potential term $\mathbf{E}$ containing $\bar{R}$)

define regulator $\mathcal{R}_k(\Box)$

$$\Gamma_k^{(2)}(\Box) + \mathcal{R}_k(\Box) \overset{\text{def.}}{=} \Gamma_k^{(2)}(\Box + R_k(\Box)),$$

where R_k is the profile function (Litim's cutoff).

flow equation

$$\int \mathrm{d}^3 x \sqrt{g}\, \partial_t f_k(\bar{R}) = \frac{1}{2} \operatorname{Tr} W(\Box)$$

heat kernel on S^3

heat kernel of scalar Laplacian $-D^2$:

$$\mathrm{Tr}\, \mathrm{e}^{-s(-D^2)} = \int \mathrm{d}^3x \sqrt{g}\, \frac{\mathrm{e}^{sR/6}}{(4\pi s)^{3/2}} \sum_{n=-\infty}^{\infty} \left(1 - \frac{12\pi^2 n^2}{Rs}\right) \mathrm{e}^{-\frac{6n^2\pi^2}{Rs}}$$

$n = 0$: local part of the heat kernel

- Visible in small R expansions.

$n \neq 0$: nonlocal/topological part of the heat kernel

- contains n-fold returning modes
- due to compactness
- "beyond all orders in R"

heat kernel on H^3

taking into account only *normalizable* eigenfunctions of $-D^2$

$$\mathrm{Tr}\,\mathrm{e}^{-s(-D^2)} = \int \mathrm{d}^3x\sqrt{g}\,\frac{\mathrm{e}^{sR/6}}{(4\pi s)^{3/2}}, \qquad R < 0$$

- no returning modes due to the non-compactness
- analytic continuation to negative R is not possible
- local (polynomial) analysis will be insensitive of the background topology

operator trace on S^3

spectral sum $\iff$ local + nonlocal heat kernel

$$\mathrm{Tr}\; W(\Box) = \sum_i D_i\, W(\lambda_i + \mathbf{E}) = \sum_{n \geq 1} n^2\, W\left((n^2 - 1)R/6 + \mathbf{E}\right)$$

local approximation: use local heat kernel only

$$\mathrm{Tr}\; W(\Box) = \int_0^\infty \mathrm{d}s\; \widetilde{W}(s) \int \mathrm{d}^3x \sqrt{g}\; \frac{\mathrm{e}^{s(R/6 - \mathbf{E})}}{(4\pi s)^{3/2}} = \int \mathrm{d}^3x \sqrt{g}\; \frac{1}{(4\pi)^{3/2}}\, Q_{3/2}\left[W\left(z - R/6 + \mathbf{E}\right)\right]$$

operator trace on H^3

using the exact heat kernel

$$\mathrm{Tr}\, W(\Box) = \int \mathrm{d}^3x \sqrt{g}\, \frac{1}{(4\pi)^{3/2}} Q_{3/2}\left[W\left(z - 1R/6 + \mathbf{E}\right)\right]$$

now: $R < 0$!

formally the analytic continuation to negative R of the local approximation on S^3.

dimensionless quantities

definition:

$$R =: k^2 r, \qquad E =: k^2 e, \qquad f_k(R) =: k^3 \varphi_k(R/k^2)$$

- flow equation: partial differential equation for $\varphi_k(r)$
- fixed functions: k stationary solutions $\iff \partial_t \varphi_k(r) = 0$
- third order equation: three parameter family of solutions

integrating out eigenmodes

optimized cutoff: $R_k(z) = (k^2 - z)\theta(k^2 - z)$

$$\mathrm{Tr}\, W(\Box) = \sum_{n\geq 1}^{N_r} A(n,\dots)\,\theta\left(1 - (n^2 - 1 + \alpha)r/6\right), \qquad e = \alpha r/6$$

- finite sum for $r > 0$
- for lowest eigenmode $(n = 1)$ to contribute $r < 6/\alpha$
- $\alpha = 1$: trace zero for $r > 6$
  $\Longrightarrow$ all fluctuations are integrated out!

singular points and pole crossing

generic ODE: $y^{(n)}(x) = f(y^{(n-1)}, \dots, y', y, x)$

r.h.s. f can have singular points

$$f(y^{(n-1)}, \dots, y', y, x) = \frac{e(y^{(n-1)}(x_0), \dots, y'(x_0), y(x_0), x_0)}{x - x_0} + \mathcal{O}\left((x - x_0)^0\right)$$

- pole crossing solution $\iff$ $e\big|_{x=x_0} = 0$ (regularity condition)
- additional boundary condition reduces number of free parameters

example

initial value problem:

$$y''(x) = -\frac{y(x)}{x-1}, \qquad y(0) = y_0,\ y'(0) = y_1$$

solutions are modified Bessel functions I_n. For y_0, y_1 generic: no global solution.

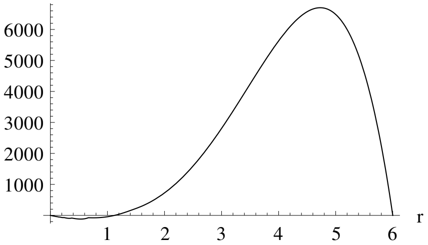
add BC y(1)=0: pole crossing solution

$$y(x; y_0) = y_0 \frac{\sqrt{1-x}\, I_1\left(2\sqrt{1-x}\right)}{I_1(2)}$$

self similarity in initial values $y(x; y_0) = \lambda^{-1} y(x; \lambda y_0)$

pole structure of flow equation

poles are zeros of the coefficient of $\varphi'''(r)$.



- three poles at $r_0 = 0$, $r_1 \approx 1$, $r_2 = 6$
- local approximation shows only two poles!
- number of fixed functions could be reduce to a finite number

numerical shooting I

expand around $r = 0$

$$\varphi(r; a_0, a_1) = a_0 + a_1 r + \sum_{n=2}^{k} a_n(a_0, a_1) r^n$$

1. fix a_2 by using regularity condition at $r = 0$

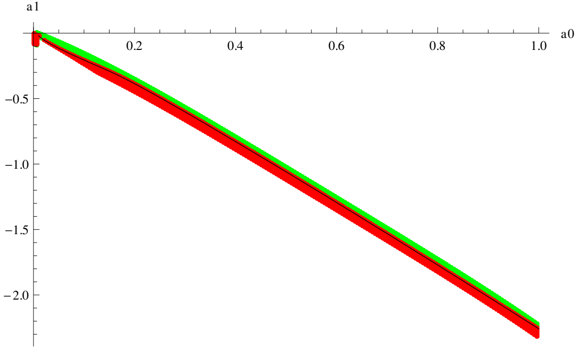
2. initial conditions for numerical integration $(\varepsilon > 0)$

$$\varphi_{\text{init}}^{(n)}(\varepsilon) = \varphi^{(n)}(\varepsilon; a_0, a_1), \qquad n = 0, 1, 2$$

3. regularity condition at r_1

$$e : (a_0, a_1) \mapsto \mathbb{R}$$

pole crossing solutions



- green: $e(a_0, a_1) > 0$
- red: $e(a_0, a_1) < 0$
- black line: $e(a_0, a_1) \approx 0$

numerical shooting II

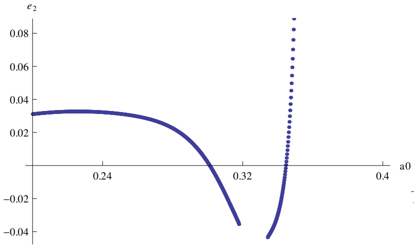
algorithm for second shooting

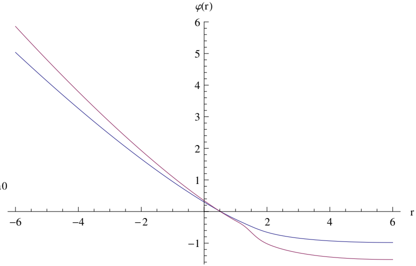
1. parametrize regular line by a_0
2. initial conditions for second numerical integration

$$\varphi_{\mathrm{init}}^{(n)}(r_{1,\mathrm{sing}} + \varepsilon) = \varphi_{\mathrm{num}}^{(n)}(r_{1,\mathrm{sing}} - \varepsilon; a_0), \qquad n = 0, 1, 2$$

3. integrate up to $r = 6$ and check regularity condition

regularity at $r = 6$





- There are two distinct zeros $\implies$ two fixed functions
- improved stability: at most three relevant directions

improved expansion: example

initial value problem:

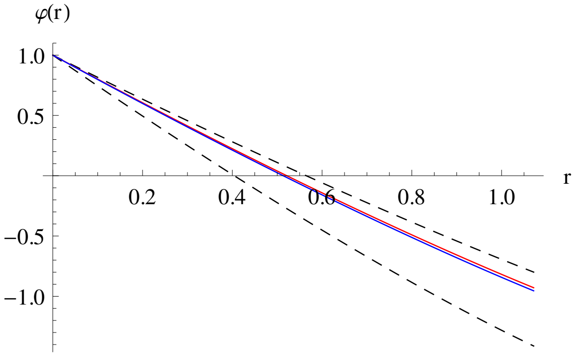
$$y''(x) = -\frac{y(x)}{x-1}$$

expansion: $y(x) = \sum_{n=0}^{k} a_n x^n$

- fixing all coefficients at $x = 0$ yields only trivial solution $\implies a_i = 0$
- improved strategy: fix a_n at $x = 0$ for $n \geq 2$ and fix a_1 at $x = 1$ $\implies$ Analytic solution can be reproduced (even self similarity)

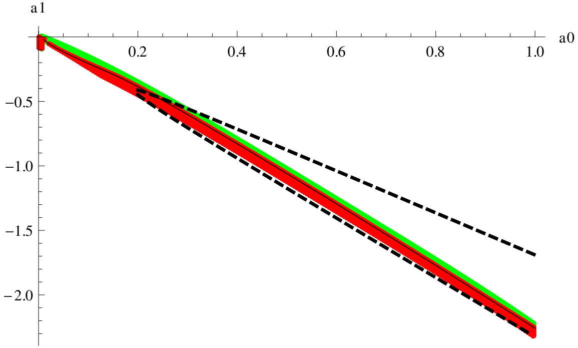
improved expansion: flow equation

Algorithm: fix a_n at $r = 0$ for $n \geq 2$ and fix a_1 at r_1
Result: there are two (a_0-dependent) solutions (dashed lines)!



regular lines

relation between a_1 and a_0 can now be computed analytically (dashed lines)



qualitative agreement!

summary

- influence of background topology
- interpretation of integrating out eigenmodes
- numerical and analytical techniques for pole crossing
- two distinct fixed functions constructed

Thank You!