Asymptotic safety and the cosmological constant

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Based on: (arXiv:1408.0276).

Introduction

- Investigate asymptotic safety considering the role of the cosmological constant.
- Quantisation of GR:

$$S = \int d^4x \sqrt{\det g_{\mu\nu}}\, \frac{1}{16\pi G}(2\Lambda - R)$$

- Einstein-Hilbert approximation

  - Two running couplings: $G, \Lambda \to G_k, \Lambda_k$

  - Degrees of freedom: dynamical graviton + non-propagating conformal fluctuations.

  - Equations of motion: $R = 4\Lambda$.

- Convexity of the effective action: $\Gamma^{(2)} > 0$

  - Wrong sign for conformal modes from the naive Wick rotation.

  - Convexity of EH not guaranteed for small curvature and momentum.

Introduction

- Wish to observe the nature of physical degrees of freedom at the level of the flow equation.
- Ensure effective action remains convex.
- Key observation: convexity is guaranteed for

$$R > 4\Lambda\,.$$

- Observable quantity :

$$\tau = G \cdot \Lambda \approx 10^{-122}$$

- What are the constraints on this number from asymptotic safety?

Physical degrees of freedom

- Field decomposition:

$$h_{\mu\nu} = h^T_{\mu\nu} + \sigma \frac{1}{d} g_{\mu\nu} + \nabla_\nu \xi_\mu + \nabla_\mu \xi_\nu + \nabla_\mu \nabla_\nu \frac{1}{\sqrt{-\nabla^2}} \psi$$

$$C_\mu = C^T_\mu + \nabla_\mu \frac{1}{\sqrt{-\nabla^2}} \eta \qquad \bar{C}_\mu = \bar{C}^T_\mu + \nabla_\mu \frac{1}{\sqrt{-\nabla^2}} \bar{\eta}$$

- Physical modes = $\{\sigma, h^T_{\mu\nu}\}$

- Gauge variant modes = $\{\xi_\mu, \psi\}$

Physical degrees of freedom

- Landau gauge —> ghost and unphysical degrees of freedom (dof) cancel.
- Six physical (i.e. gauge invariant) degrees of freedom.
- How many propagate?
- Two polarisations of the graviton.
- Additional negative dof from the Jacobian in the functional measure:

$$J_0 = \sqrt{\det{}'' \left[-\nabla^2 - \frac{R}{d-1} \right]}, \quad J_1 = \sqrt{\det{}' \left[-\nabla^2 - \frac{R}{d} \right]},$$

- Acting on scalars and transverse vectors.

Physical degrees of freedom

- Number of propagating comes from physical dof minus Jacobians :

$$\frac{1}{2}d(d-1) - d = \frac{1}{2}d(d-3)$$

- Hessians (d=4): $\Delta \equiv 16\pi G_k \, \Gamma_k^{(2)}$

$$\Delta_T = \Delta_2 + 2\left(\frac{R}{4} - \Lambda_k\right), \quad \Delta_\sigma = \Delta_0 + \frac{4}{3}\left(\frac{R}{4} - \Lambda_k\right),$$

$$\Delta_0 = -\nabla^2 - \frac{R}{3}, \quad \Delta_1 = -\nabla^2 - \frac{R}{4},$$

Conformal modes

- Mottola and Mazur '90, Wick rotation of conformal modes implied by de Witt's supermetric :

$$\Delta_0 > 0 \implies \sigma \to i\sigma\,,$$
$$\Delta_0 = 0 \implies \text{Conformal Killing modes (unphysical)}\,.$$
$$\Delta_0 < 0 \implies \sigma \to \sigma\,.$$

- On shell cancellations, i.e. for $R = 4\Lambda$, occur between conformal fluctuations and the Jacobian:

$$\Delta_\sigma = \Delta_0 + \frac{4}{3}\left(\frac{R}{4} - \Lambda_k\right) \qquad\qquad \Delta_0 = -\nabla^2 - \frac{R}{3}$$

- This leaves the constant mode:

$$-\Delta_\sigma \sigma = a_- \sigma \qquad\qquad a_- = +\frac{R}{3} - \frac{4}{3}\left(\frac{R}{4} - \Lambda_k\right)$$

Flow equation

- Terms arise from the EH action and from the functional measure:

$$\partial_t \Gamma_k = \frac{1}{2}\,\mathrm{Tr}\left[\frac{\partial_t \mathcal{R}_{T,k}}{Z_k \Delta_T + \mathcal{R}_{T,k}}\right] + \frac{1}{2}\mathrm{Tr}''\left[\frac{\partial_t \mathcal{R}_{\sigma,k}}{Z_k \Delta_\sigma + \mathcal{R}_{\sigma,k}}\right] + \frac{1}{2}\left[\frac{\partial_t \mathcal{R}_{-,k}}{Z_k a_- + \mathcal{R}_{-,k}}\right]$$
$$-\frac{1}{2}\mathrm{Tr}''\left[\frac{\partial_t \mathcal{R}_{0,k}}{Z_k \Delta_0 + \mathcal{R}_{0,k}}\right] - \frac{1}{2}\mathrm{Tr}'\left[\frac{\partial_t \mathcal{R}_{1,k}}{Z_k \Delta_1 + \mathcal{R}_{1,k}}\right]$$

- Regroup as physical degrees of freedom:
  - Two polarisations of the graviton
  - Topological conformal mode

$$\partial_t \Gamma_k = \frac{1}{2}\,\mathrm{Tr}\left[\frac{\partial_t \mathcal{R}_{T,k}}{Z_k \Delta_T + \mathcal{R}_{T,k}}\right] - \frac{1}{2}\mathrm{Tr}'\left[\frac{\partial_t \mathcal{R}_{1,k}}{Z_k \Delta_1 + \mathcal{R}_{1,k}}\right]$$

$$+\frac{1}{2}\mathrm{Tr}''\left[\frac{\partial_t \mathcal{R}_{\sigma,k}}{Z_k \Delta_\sigma + \mathcal{R}_{\sigma,k}}\right] - \frac{1}{2}\mathrm{Tr}''\left[\frac{\partial_t \mathcal{R}_{0,k}}{Z_k \Delta_0 + \mathcal{R}_{0,k}}\right] + \frac{1}{2}\left[\frac{\partial_t \mathcal{R}_{-,k}}{Z_k a_- + \mathcal{R}_{-,k}}\right]$$

Convexity

- For the regulated theory the convexity condition is given by:

$$\Gamma_k^{(2)} + \mathcal{R}_k > 0$$

- Here we take a regulator of the form:

$$\mathcal{R}_k = \frac{1}{16\pi G_k} R_k(z)\,,$$

- Poles in the propagator imply that we violate convexity.
- These can arise in curvature expansions for which at vanishing curvature we have:

$$P_\sigma(R=0) \equiv -\nabla^2 - \frac{4}{3}\Lambda_k + R_{\sigma,k} \stackrel{!}{=} 0\,,$$

$$P_T(R=0) \equiv -\nabla^2 - 2\Lambda_k + R_{T,k} \stackrel{!}{=} 0\,.$$

Convexity

- Convexity is ensured if we take $R > 4\Lambda$.
- Now consider a regulator which depends on the full Hessian (Type III)

$$\mathcal{R}_k = \frac{1}{16\pi G_k} R_k(\Delta)$$

with

$$\Delta_T = \Delta_2 + 2\left(\frac{R}{4} - \Lambda_k\right), \quad \Delta_\sigma = \Delta_0 + \frac{4}{3}\left(\frac{R}{4} - \Lambda_k\right),$$

- Positivity of the argument of the regulator then requires

$$R > 4\Lambda.$$

Summary

- We have made two observation relating to the on shell condition for the curvature:

1. Cancellations between conformal fluctuations and the functional measure occur on shell
2. For curvature greater than the on shell value convexity is guaranteed.

- We now want to employ a simple approximation where the implications of these can be observed.

Heat kernel expansion

- Since we are interested in effects around the on shell condition we want an approximation sensitive to effects

$$R \sim 4\Lambda_k \, .$$

- We therefore shall utilise the type III regulator while using the heat kernel expansion for the full Hessian:

$$\int ds \, \mathrm{Tr}[e^{-\Delta s}] \, \tilde{f}(\tau) \approx \frac{1}{(4\pi)^{\frac{d}{2}}} \sum_{n=0}^{\infty} Q_{\frac{d}{2}-n}[f] A_n(R, \Lambda_k)$$

- Comments

$$\Delta \equiv 16\pi G_k \, \Gamma_k^{(2)}$$

- This expansion naturally treats the curvature and cosmological constant on the same footing.

- Convexity i.e. positivity of Δ is required by the anti-Laplace transform, hence the argument of the regulator is positive.

Heat kernel expansion

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:

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- Our approximation will be to truncate the heat kernel expansion at n=1:

  - Expect to give good UV limit while the curvature remains close to the on shell value.

  -Convexity is built in.

Results

- We can now evaluate the traces to obtain the beta functions for the dimensionless couplings:

$$\beta_g = g\left(2+\frac{g\left(-438g\Phi_2\hat{\Phi}_1+\Phi_1\left(752g\lambda\hat{\Phi}_1+99\left(6\pi+17g\hat{\Phi}_2\right)\right)\right)}{9\left(6\pi+17g\hat{\Phi}_2\right)\left(-4\pi+11g\tilde{\Phi}_1\right)+2g\hat{\Phi}_1\left(376g\lambda\tilde{\Phi}_1-3\left(50\pi\lambda+73g\tilde{\Phi}_2\right)\right)}\right),$$

$$\beta_\lambda = -2\lambda-\frac{9g\left(2\pi\left(\lambda\Phi_1+6\Phi_2\right)+33g\left(-\Phi_2\tilde{\Phi}_1+\Phi_1\tilde{\Phi}_2\right)\right)}{-216\pi^2+6g\pi\left(-50\lambda\hat{\Phi}_1-102\hat{\Phi}_2+99\tilde{\Phi}_1\right)+g^2\left(752\lambda\hat{\Phi}_1\tilde{\Phi}_1+1683\hat{\Phi}_2\tilde{\Phi}_1-438\hat{\Phi}_1\tilde{\Phi}_2\right)}.$$

- Here the beta functions depend on regulator dependent numbers

$$\Phi_m=\int_0^\infty dz z^{m-1}\frac{\dot{R}_1(z)}{z+R_1(z)},\quad \tilde{\Phi}_m=\int_0^\infty dz z^{m-1}\frac{R_1(z)}{z+R_1(z)},\quad \hat{\Phi}_m=\int_0^\infty dz z^{m-1}\frac{R_1'(z)}{z+R_1(z)},$$

UV fixed point

- Fixed points for positive couplings

$$g_* = \frac{576\pi}{208\Phi_1 + 416\tilde{\Phi}_1 + 73\left(-17\hat{\Phi}_2 + \sqrt{\left(8\Phi_1 + 17\hat{\Phi}_2 + 16\tilde{\Phi}_1\right)^2 - 96\hat{\Phi}_1\left(\Phi_2 + 2\tilde{\Phi}_2\right)}\right)},$$

$$\lambda_* = \frac{8\Phi_1 + 17\hat{\Phi}_2 + 16\tilde{\Phi}_1 - \sqrt{\left(8\Phi_1 + 17\hat{\Phi}_2 + 16\tilde{\Phi}_1\right)^2 - 96\hat{\Phi}_1\left(\Phi_2 + 2\tilde{\Phi}_2\right)}}{32\hat{\Phi}_1}.$$

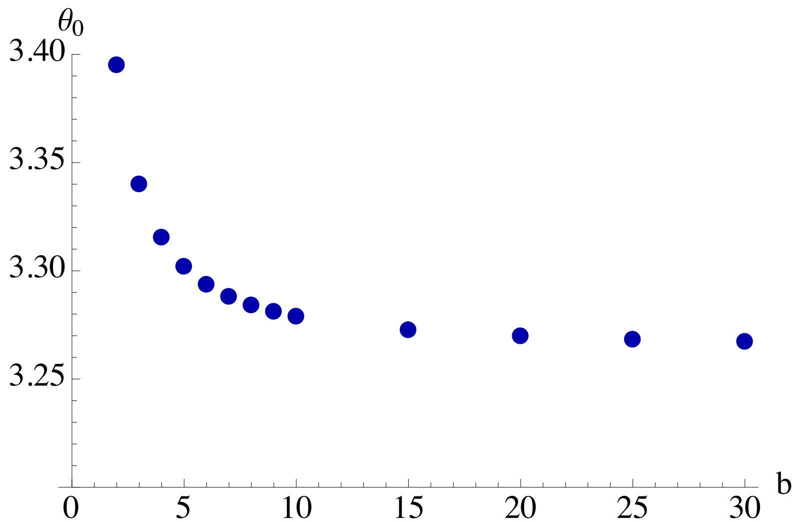
- Positive and real fixed points for all regulator functions.
- Critical exponents are real and relevant:

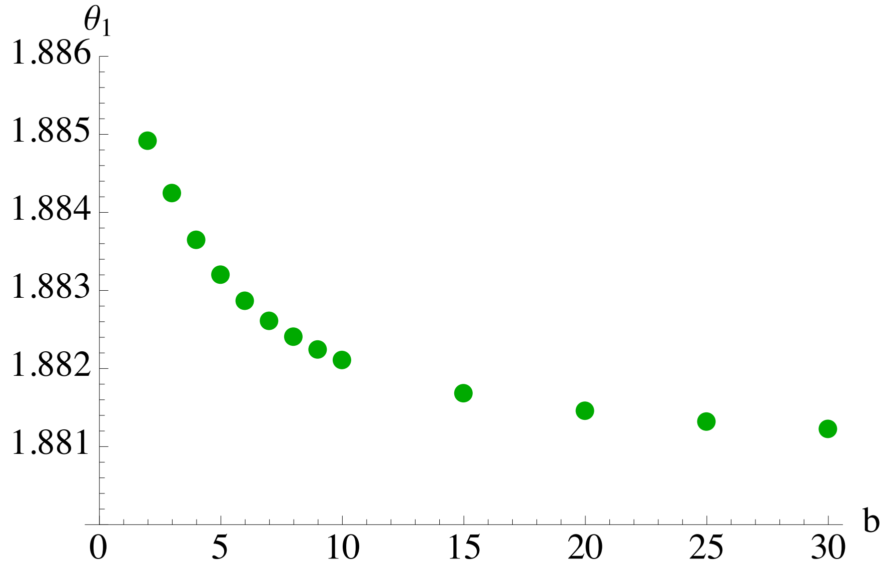
$$\theta_0 \overset{\text{opt}}{\approx} 3.35126\,, \quad \theta_1 \overset{\text{opt}}{\approx} 1.87582\,.$$

Using the Litim's optimised cutoff $R_k(z) = (k^2 - z)\Theta(k^2 - z)$.

UV fixed point

- Regulator dependence:





$$R_k^{\exp}(z) = k^2 \frac{1}{2\exp\left[c\frac{z^b}{k^{2b}}\right] - 1}$$

UV fixed point at one loop

- One loop flow: $\partial_t \Gamma_k^{\mathrm{one-loop}} = \frac{1}{2} \mathrm{STr} \left[\frac{\partial_t \mathcal{R}_k}{S^{(2)} + \mathcal{R}_k} \right]$

$$\beta_g = 2g - \frac{11 g^2 \Phi_1}{4\pi}\,, \qquad \beta_\lambda = \frac{-24\pi\lambda + g\lambda\Phi_1 + 6g\Phi_2}{12\pi}\,.$$

- Fixed point:

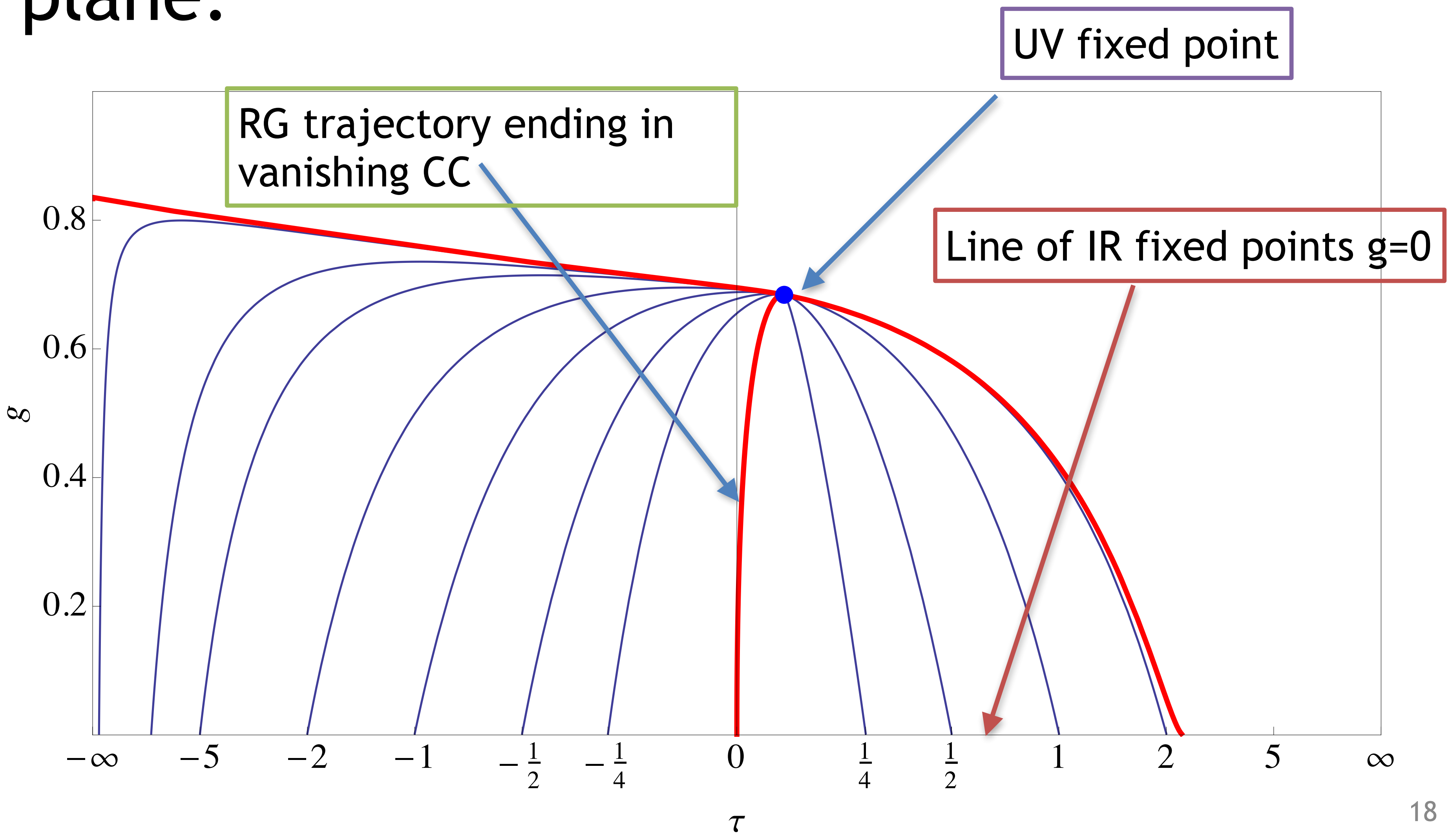
$$g_* = \frac{8\pi}{11\Phi_1}\,, \qquad \lambda_* = \frac{3\Phi_2}{16\Phi_1}\,,$$

- Critical exponents are regulator independent(!):

$$\theta_0 = 2\,, \quad \theta_1 = \frac{64}{33} \approx 1.939\,.$$

Globally safe trajectories

- RG trajectories in the $\{\tau = G\Lambda, g\}$ plane:



Globally safe trajectories

- By enforcing convexity RG trajectories can have classical scaling limits for $k \to 0$ by reaching the line of IR fixed points:

$$g_* = 0\,, \quad \tau = \text{const} = G_N \cdot \Lambda\,, \quad G_k = G_N\,.$$

- Globally safe trajectories exist for all values of the cosmological constant

$$\Lambda_0 G_0 < \tau_{\max}\,, \quad \tau_{\max} \sim 1\,.$$

Conformally reduced theory

- New approximation keep just conformal fluctuations and the scalar Jacobian.
- Conformal fluctuations remain topological.
- If we disregard the Jacobian we get a propagating scalar degree of freedom not present in GR.

$$\partial_t \Gamma_k = \frac{1}{2}\mathrm{Tr}''\left[\frac{\partial_\mathrm{t}\mathcal{R}_{\sigma,\mathrm{k}}}{\mathrm{Z_k}\Delta_\sigma + \mathcal{R}_{\sigma,\mathrm{k}}}\right] - \frac{1}{2}\mathrm{Tr}''\left[\frac{\partial_\mathrm{t}\mathcal{R}_{0,\mathrm{k}}}{\mathrm{Z_k}\Delta_0 + \mathcal{R}_{0,\mathrm{k}}}\right]$$

Conformally reduced theory

- First we include only the conformal modes without the Jacobian.

$$\partial_t \Gamma_k = \frac{1}{2}\mathrm{Tr}''\left[\frac{\partial_t \mathcal{R}_{\sigma,k}}{Z_k \Delta_\sigma + \mathcal{R}_{\sigma,k}}\right]$$

- Here we find the regulator dependence is strong and the critical exponents can change sign depending on the regulator function.

$$\theta_0 \overset{\mathrm{opt}}{\approx} 1.53784\,, \qquad \theta_1 \overset{\mathrm{opt}}{\approx} -19.6375$$

$$\theta_0 \approx 367.403\,,\;\; \theta_1 \approx 1.48858 \;\; \text{exponential cutoff}(\mathrm{b}=2)$$

- This indicates that this is a bad approximation.

Conformally reduced theory

- Including the Jacobian the beta functions are given by:

$$\beta_\lambda = \lambda \left(-2 - \frac{3g\pi\Phi_1}{-\left(3\pi + g\hat{\Phi}_2\right)\left(6\pi - g\tilde{\Phi}_1\right) + 2g\lambda\hat{\Phi}_1\left(-5\pi + g\tilde{\Phi}_1\right)} \right)$$

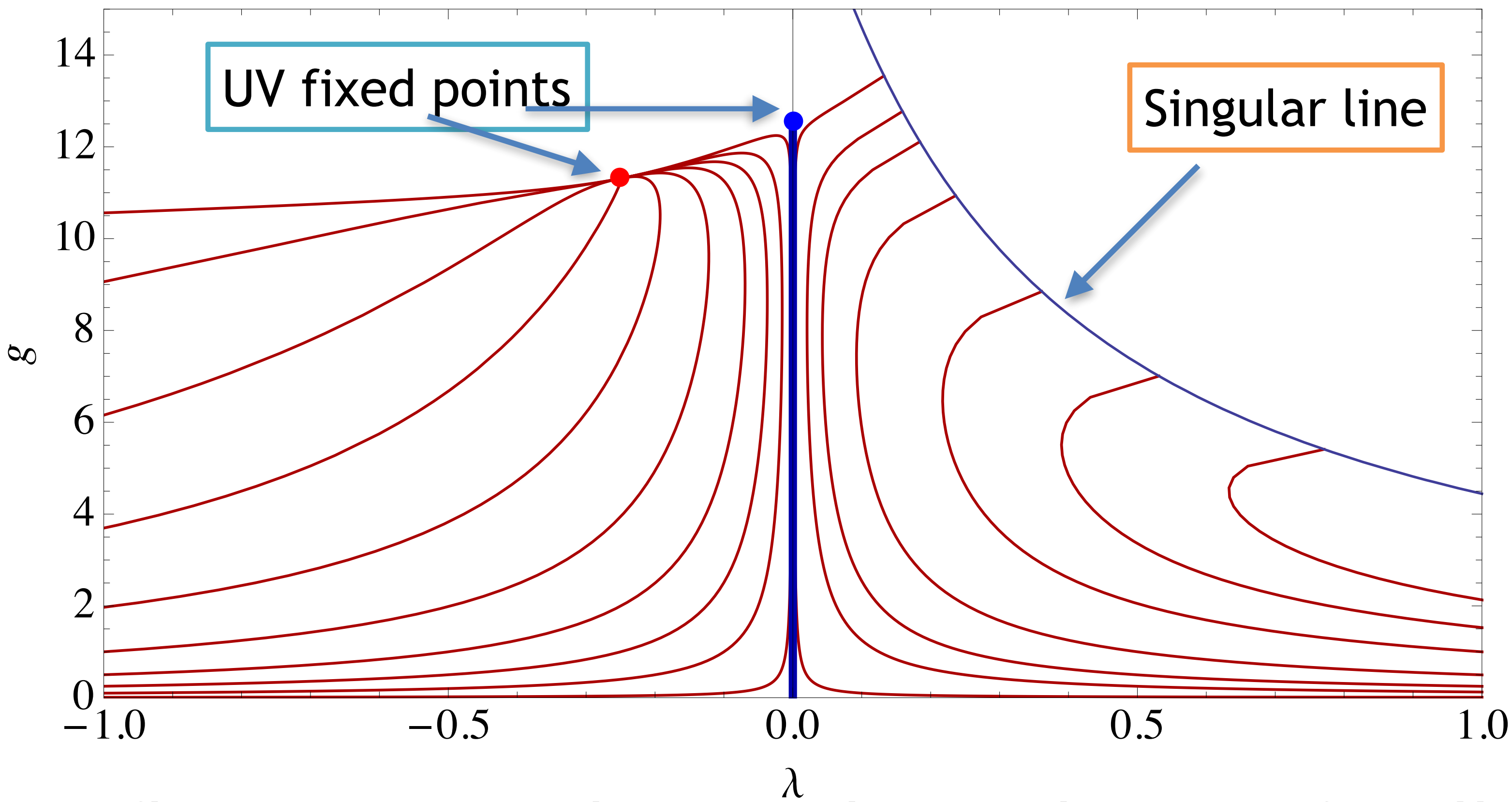
$$\beta_g = g \left(2 + \frac{g\Phi_1\left(3\pi + 2g\lambda\hat{\Phi}_1 + g\hat{\Phi}_2\right)}{-\left(3\pi + g\hat{\Phi}_2\right)\left(6\pi - g\tilde{\Phi}_1\right) + 2g\lambda\hat{\Phi}_1\left(-5\pi + g\tilde{\Phi}_1\right)} \right) .$$

- The beta function for lambda vanishes identically at

$$\lambda = 0.$$

Flow diagram

- Flow for driven by conformal modes with non-trivial Jacobian.



- RG flow contains phase with vanishing CC for all scales.
- UV fixed point for negative CC.
- No UV completion for positive CC.

Critical exponents

- At the $\Lambda = 0$ UV fixed point there is only one relevant direction.

$$\theta_0 = 2 + \frac{4\tilde{\Phi}_1}{\Phi_1} \overset{\text{opt}}{=} 3\,, \qquad \theta_{\text{irr}} = \frac{8\hat{\Phi}_2}{\Phi_1 + 4\hat{\Phi}_2 + 2\tilde{\Phi}_1} \overset{\text{opt}}{=} -4\,.$$

- The relevant exponent corresponds to the critical exponent

$$\nu = 1/\theta_0 \qquad\qquad \xi k = \frac{1}{(g_* - g)^\nu}$$

- Our result is in agreement with lattice calculations of quantum Regge calculus by Hamber:

Continuum FRG:

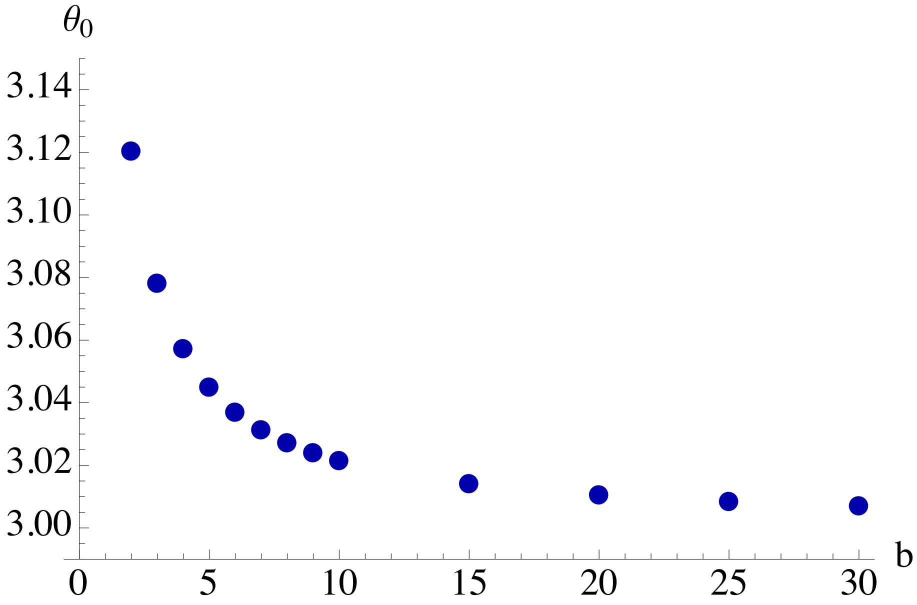
$$\nu \overset{\text{opt}}{=} 1/3$$

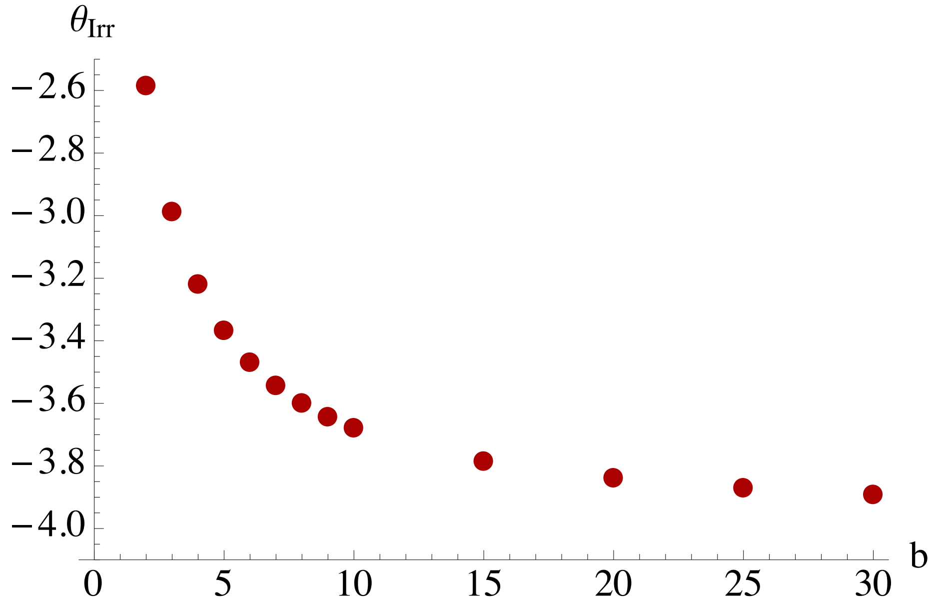
Lattice theory:

$$\nu \approx 0.3354$$

Regulator dependence

- Critical exponents for the exponential cutoff:





- The relevant exponent has a very mild regulator dependence

Critical exponent in the epsilon expansion

- Can generalise this calculation to arbitrary dimension
- Using the optimised cutoff the relevant critical exponent is then given by:

$$\theta_0 \equiv 1/\nu \stackrel{\text{opt}}{=} 2d + \frac{4}{d} - 6 = \frac{2\epsilon(1+\epsilon)}{2+\epsilon}\,,$$

where $\epsilon = d - 2\,.$

- We note that four dimensions lies on the radius of convergence of the epsilon expansion

$$1/\nu = \epsilon + \frac{\epsilon^2}{2} - \frac{\epsilon^3}{4} + \frac{\epsilon^4}{8} + \dots\,.$$

- Truncating at order ϵ^2 and setting d=4 we can compare with the two loop result of Aida and Kitazawa '97:

FRG: $1/\nu = 4$ Two loop: $1/\nu \approx 4.4$

Conclusions

- Identified propagating and topological contributions to the flow equation with contributions from the action and functional measure.
- Convexity is ensured by taking the curvature to be greater than its on shell value.
- Approximation obtained by a truncation of the heat kernel expansion with a built in convexity condition.
- Obtained real critical exponents at a UV fixed points leading to to classical IR limit.
- New conformally reduced approximation which keeps the topological nature of conformal fluctuations.
- Fixed point for vanishing vacuum energy at all scales.
- Critical exponents found in good agreement with lattice and perturbation theory.