Global surpluses of spin-base invariant Fermions

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[arXiv:1502.00918 [hep-th]] and work in preparation

Motivation

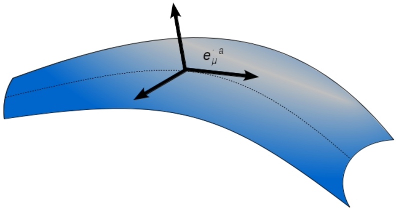
- aim at a theory of quantized matter and quantized forces
- matter is composed of fermions $\Rightarrow \mathcal{D}\psi\mathcal{D}\bar{\psi}$ ✓
- SM forces mediated via gauge field $\Rightarrow \mathcal{D}\mathcal{A}_\mu{}^a$ ✓
- gravity encoded in spacetime curvature $\Rightarrow \mathcal{D}g_{\mu\nu}$, $\mathcal{D}e_\mu{}^a$, $\mathcal{D}$? ✗

$\Rightarrow$ need to describe fermions in curved spacetime

$\Rightarrow$ vielbein formalism is mandatory [Weyl '29, DeWitt '65]

$$e_\mu{}^a e_\nu{}^b \eta_{ab} = g_{\mu\nu}, \quad \nabla_\mu e_\nu{}^a = 0 \qquad ?$$

hidden spin-base invariance:



- assume a vielbein $e_\mu{}^a$ with local $\mathrm{SO}(d-1,1)$ symmetry
- independence of the tangential spaces of every point of the manifold $\Rightarrow$ reason for locality of $\mathrm{SO}(d-1,1)$
- covariant derivative: $D_\mu e_\nu{}^a + \omega_\mu{}^a{}_b e_\nu{}^b = 0$

- fermions need Dirac structure
  $\Rightarrow$ Clifford algebra in tangential space:
  $$\{\gamma_{(\mathrm{f})a}, \gamma_{(\mathrm{f})b}\} = 2\eta_{ab}\mathrm{I}$$

- independence of the tangential spaces
  $\Rightarrow$ local changes of the Dirac matrix representation:
  $$\gamma_{(\mathrm{f})a} = \gamma_{(\mathrm{f})a}(x) \qquad \text{(hidden spin-base invariance)}$$

- behavior under spin-base transformations $\mathcal{S} \in \mathrm{SL}(d_\gamma, \mathbb{C})$:
  [Schrödinger '32, Bargmann '32, Pauli '36]
  $$\gamma_{(\mathrm{f})a} \to \mathcal{S}\gamma_{(\mathrm{f})a}\mathcal{S}^{-1}$$

- standard spin connection for constant ${\gamma_{(\mathrm{f})}}_a$

$$D_\mu {e_\nu}^a + {\omega_\mu}^a{}_b {e_\nu}^b = 0 \Leftrightarrow D_\mu {e_\nu}^a {\gamma_{(\mathrm{f})}}_a + [\hat{\Gamma}_\mu, {e_\mu}^a {\gamma_{(\mathrm{f})}}_a] = 0$$

where $\quad \hat{\Gamma}_\mu = \frac{1}{8} {\omega_\mu}^{ab} [{\gamma_{(\mathrm{f})}}_a, {\gamma_{(\mathrm{f})}}_b]$

- behavior of spin connection $\hat{\Gamma}_\mu$ under spin-base transformations

$$\hat{\Gamma}_\mu \to \mathcal{S} \hat{\Gamma}_\mu \mathcal{S}^{-1} - (\partial_\mu \mathcal{S}) \mathcal{S}^{-1}$$

- it seems we first need ${e_\mu}^a$ and then ${\gamma_{(\mathrm{f})}}_a$ together with $\mathcal{S}$ to calculate the spin connection $\hat{\Gamma}_\mu$ for an arbitrary spin-base

- BUT: $\hat{\Gamma}_\mu$ can be derived from

$$D_\mu \gamma_\nu + [\hat{\Gamma}_\mu, \gamma_\nu] = 0, \quad \mathrm{tr}\, \hat{\Gamma}_\mu = 0$$

  for arbitrary spin-bases $\Rightarrow$ relevant objects are the γ_μ

- introduction of vielbein is artificial split of γ_μ into $e_\mu{}^a$ and $\gamma_{(\mathrm{f})\,a}$

- Dirac matrices are more general (global surpluses)
  [Kofink '49, Brill and Wheeler '57, Unruh '74, Finster et al. '99, Casals et al. '13, Gies and Lippoldt '13, Christiansen et al. '15]

$\Rightarrow$ formalize construction without introduction of vielbein

Spin-base invariance

Dirac structure:

- Clifford algebra (irreducible representation):
  $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathrm{I}, \quad \gamma_\mu \in \mathbb{C}^{d_\gamma \times d_\gamma}, \quad d_\gamma = 2^{\lfloor d/2 \rfloor}$

- Dirac conjugation with spin metric h:
  $\bar{\psi} = \psi^\dagger h, \quad |\det h| = 1$

perform coordinate transformation $\gamma_\mu \to \gamma'_\mu$

$$\{\gamma'_\mu, \gamma'_\nu\} = 2g'_{\mu\nu}\mathrm{I} = 2\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\lambda}{\partial x'^\nu}g_{\rho\lambda}\mathrm{I} = \{\frac{\partial x^\rho}{\partial x'^\mu}\gamma_\rho, \frac{\partial x^\lambda}{\partial x'^\nu}\gamma_\lambda\}$$

most general solution $\gamma'_\mu = \pm\frac{\partial x^\rho}{\partial x'^\mu}\mathcal{S}\gamma_\rho\mathcal{S}^{-1}$, $\mathcal{S} \in \mathrm{SL}(d_\gamma, \mathbb{C})$

have two independent coordinate transformations

- diffeomorphisms (change of spacetime base):
  $\gamma_\mu \to \gamma'_\mu = \frac{\partial x^\rho}{\partial x'^\mu}\gamma_\rho$
  $\psi \to \psi' = \psi$
  $h \to h' = h$

- spin base transformations (change of spin base):
  $\gamma_\mu \to \gamma'_\mu = \pm\mathcal{S}\gamma_\mu\mathcal{S}^{-1}$
  $\psi \to \mathcal{S}\psi$
  $h \to \pm(\mathcal{S}^\dagger)^{-1}h\mathcal{S}^{-1}$

covariant derivative:

- linearity:
  $$\nabla_\mu(\psi_1 + \psi_2) = \nabla_\mu\psi_1 + \nabla_\mu\psi_2$$
- product rule:
  $$\nabla_\mu\psi\bar{\psi} = (\nabla_\mu\psi)\bar{\psi} + \psi(\nabla_\mu\bar{\psi})$$
- spin metric compatibility:
  $$\nabla_\mu\bar{\psi} = \overline{\nabla_\mu\psi} \qquad \Big| \quad D_\mu T_\nu = (D_\mu T^\rho)g_{\rho\nu}$$
- coordinate covariance:
  $$\nabla_\mu(\bar{\psi}\gamma^\nu\psi) = D_\mu(\bar{\psi}\gamma^\nu\psi) \equiv \partial_\mu(\bar{\psi}\gamma^\nu\psi) + \Gamma^\nu_{\mu\kappa}(\bar{\psi}\gamma^\kappa\psi),$$
  $$\Gamma^\nu_{\mu\kappa} = \left\{ {\nu \atop \mu\kappa} \right\} + K^\nu{}_{\mu\kappa}, \quad \left\{ {\nu \atop \mu\kappa} \right\} = \tfrac{1}{2}g^{\nu\rho}(\partial_\mu g_{\kappa\rho} + \partial_\kappa g_{\mu\rho} - \partial_\rho g_{\mu\kappa})$$

reality of action:

- mass term:
  $(\bar{\psi}\psi)^* = \bar{\psi}\psi$

- kinetic term:
  $\int \mathrm{d}^d x \sqrt{-g}\ (\bar{\psi}\nabla\!\!\!/\,\psi)^* = \int \mathrm{d}^d x \sqrt{-g}\ \bar{\psi}\nabla\!\!\!/\,\psi, \quad \nabla\!\!\!/\,\psi = \gamma^\mu \nabla_\mu \psi$

Spin metric and spin connection

- construct spin metric h and spin covariant derivative ∇_μ
- spin metric is implicitly determined by the γ_μ

Spin metric

$$\gamma_\mu^\dagger = -h\gamma_\mu h^{-1}, \quad h^\dagger = -h, \quad |\det h| = 1$$

spin covariant derivative:

$$\nabla_\mu \psi = \partial_\mu \psi + \hat{\Gamma}_\mu \psi + \Delta\Gamma_\mu \psi \ \Big| \ D_\mu v^\alpha = \partial_\mu v^\alpha + \left\{ {\alpha \atop \mu\beta} \right\} v^\beta + K^\alpha{}_{\mu\beta} v^\beta$$

canonical part of the spin connection $\hat{\Gamma}_\mu$

$$D_{(\mathrm{LC})_\mu} \gamma^\nu = \partial_\mu \gamma^\nu + \left\{ {\nu \atop \mu\kappa} \right\} \gamma^\kappa = -[\hat{\Gamma}_\mu, \gamma^\nu], \quad \mathrm{tr}\, \hat{\Gamma}_\mu = 0$$

for example in $d = d_\gamma = 4$:

$$\hat{\Gamma}_\mu = p_\mu \gamma_* + v_\mu{}^\alpha \gamma_\alpha + a_\mu{}^\alpha \gamma_* \gamma_\alpha + t_\mu{}^{\alpha\beta} [\gamma_\alpha, \gamma_\beta],$$

$$\gamma_* = -\frac{\mathrm{i}}{4!} \sqrt{-g}\, \varepsilon_{\mu_1 \dots \mu_4} \gamma^{\mu_1} \dots \gamma^{\mu_4}$$

$$p_\mu = \frac{1}{32} \mathrm{tr}(\gamma_* \gamma_\alpha \partial_\mu \gamma^\alpha), \quad v_\mu{}^\alpha = \frac{1}{48} \mathrm{tr}([\gamma^\alpha, \gamma_\beta] \partial_\mu \gamma^\beta),$$

$$a_\mu{}^\alpha = \frac{1}{8} \mathrm{tr}(\gamma_* \partial_\mu \gamma^\alpha), \quad t_{\mu\alpha}{}^\beta = -\frac{1}{32} \mathrm{tr}(\gamma_\alpha \partial_\mu \gamma^\beta) - \frac{1}{8} \left\{ {\beta \atop \mu\alpha} \right\}$$

- spin torsion $\Delta\Gamma_\mu$:
  $0 = [\Delta\Gamma_\mu, \gamma^\mu], \quad \Delta\Gamma_\mu = -h^{-1}\Delta\Gamma_\mu^\dagger h$

- for example in $d = d_\gamma = 4$:
  spin torsion carries 45 real parameters, but only 11 remain within the Dirac operator $\nabla\!\!\!/$:

$$\bar\psi\gamma^\mu\Delta\Gamma_\mu\psi = \mathscr{M}\bar\psi\psi - \mathscr{A}_\mu\bar\psi\mathrm{i}\gamma_*\gamma^\mu\psi - \mathscr{F}_{\mu\nu}\bar\psi\tfrac{\mathrm{i}}{4}[\gamma^\mu,\gamma^\nu]\psi$$

$\mathscr{M}$: mass/scalar field

$\mathscr{A}_\mu$: axial vector field

$\mathscr{F}_{\mu\nu}$: anti-symmetric tensor field

Intermission

Questions?

Path integral

- can construct action S if γ^μ are known
- naive way: $\int \mathcal{D}\gamma \, \mathrm{e}^{\mathrm{i}S}$ (and fermions, gauge fields, ...)
- but the Clifford algebra $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathrm{I}$ prohibits arbitrary variations of γ_μ
  $\Rightarrow$ determine degrees of freedom from Clifford algebra

Weldon theorem [Weldon '01] ($d = 4$), [Lippoldt '15] ($d \geq 2$)

$$\delta\gamma_\mu = \frac{1}{2}(\delta g_{\mu\nu})\gamma^\nu + [\delta\mathcal{S}_\gamma, \gamma_\mu], \ \mathrm{tr}\,\delta\mathcal{S}_\gamma = 0$$

$\delta g_{\mu\nu}$: metric fluctuations

$\delta\mathcal{S}_\gamma$: spin-base fluctuations, $\mathrm{SL}(d_\gamma, \mathbb{C})$

want to show that the $\mathcal{S}_\gamma$ part factorizes

ingredients for proof:

- action as a functional of the fermions ψ, $\bar{\psi}$, the metric $g_{\mu\nu}$ and the spin-base $\mathcal{S}_\gamma$: $S = S[\psi, \bar{\psi}, g; \mathcal{S}_\gamma]$
- spin-base invariance of S:
  $S[\psi, \bar{\psi}, g; \mathcal{S}_\gamma] \to S[\mathcal{S}\psi, \bar{\psi}\mathcal{S}^{-1}, g; \mathcal{S}'_\gamma] \equiv S[\psi, \bar{\psi}, g; \mathcal{S}_\gamma]$
- spin-base invariance of the measure $\mathcal{D}\psi\mathcal{D}\bar{\psi}$:
  $\mathcal{D}\psi\mathcal{D}\bar{\psi} \to \mathcal{D}(\mathcal{S}\psi)\mathcal{D}(\bar{\psi}\mathcal{S}^{-1}) = \mathcal{D}\psi\mathcal{D}\bar{\psi}$

- study expectation value of an operator $\hat{O}(\psi, \bar{\psi}, g; \mathcal{S}_\gamma)$, which is a scalar under spin-base transformations

- consider integration over fermions and metric

$$\left\langle \hat{O}(\psi, \bar{\psi}, g; \mathcal{S}_\gamma) \right\rangle = \int \mathcal{D}g \mathcal{D}\psi \mathcal{D}\bar{\psi}\, \hat{O}(\psi, \bar{\psi}, g; \mathcal{S}_\gamma) \mathrm{e}^{\mathrm{i}S[\psi, \bar{\psi}, g; \mathcal{S}_\gamma]}$$

- with spin-base invariance we find

$$\left\langle \hat{O}(\psi, \bar{\psi}, g; \mathcal{S}_\gamma) \right\rangle \equiv \left\langle \hat{O}(\psi, \bar{\psi}, g; \mathcal{S}'_\gamma) \right\rangle$$

$\Rightarrow$ integration over $\mathrm{SL}(d_\gamma, \mathbb{C})$ is trivial

in practice:
fix spin base for purely metric-based quantization

justification for:
common use of the vielbein as a function of the metric
(without keeping Jacobian from variable transformation $\mathcal{D}e \to \mathcal{D}g$)

- look for *simplest* explicit gauge fixing of the γ_μ for a given metric $g_{\mu\nu}$, w.r.t. a background metric $\bar{g}_{\mu\nu}$:
  $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$
  $$\gamma_\mu(g) = \bar{\gamma}_\mu + \sum_{n=1}^{\infty} \frac{1}{n!} \left. \frac{\partial^n \gamma_\mu(g)}{\partial g_{\rho_1\rho_2} \ldots \partial g_{\rho_{2n-1}\rho_{2n}}} \right|_{g=\bar{g}} h_{\rho_1\rho_2} \ldots h_{\rho_{2n-1}\rho_{2n}}$$
- using the Weldon theorem:
  $$\frac{\partial \gamma_\mu(g)}{\partial g_{\rho_1\rho_2}} = \frac{1}{2}\delta_S{}^{\rho_1\rho_2}_{\mu\ \nu}\gamma^\nu(g) + [G^{\rho_1\rho_2}(g), \gamma_\mu(g)]$$
- tuning of the spin base to make expansion coefficients *simple*
- the symmetric part and the commutator part are independent $\Rightarrow$ possible to eliminate the commutator part:
  $$\left. \frac{\partial^n \gamma_\mu(g)}{\partial g_{\rho_1\rho_2} \ldots \partial g_{\rho_{2n-1}\rho_{2n}}} \right|_{g=\bar{g}} \sim \bar{\gamma}^\kappa$$

- make new ansatz
  $\gamma_\mu(g) = \mathcal{B}_{\mu\nu}(g,\bar{g})\bar{\gamma}^\nu, \quad$ with $\mathcal{B}_{\mu\nu}(g,\bar{g}) = \mathcal{B}_{\nu\mu}(g,\bar{g})$

- plugging into the Clifford algebra
  $$2g_{\mu\nu}\mathrm{I} = \{\mathcal{B}_{\mu\rho}\bar{\gamma}^\rho, \mathcal{B}_{\nu\kappa}\bar{\gamma}^\kappa\} = 2\mathcal{B}_{\mu\rho}\bar{g}^{\rho\kappa}\mathcal{B}_{\kappa\nu}\mathrm{I}$$

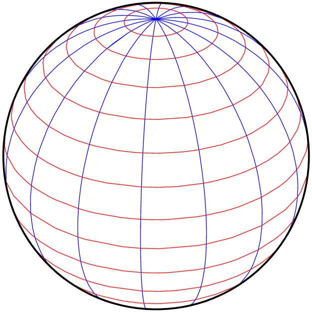
- in matrix form we have
  $$g = \mathcal{B}\bar{g}^{-1}\mathcal{B} = \bar{g}(\bar{g}^{-1}\mathcal{B})^2$$

- and therefore we find
  $$\mathcal{B} = \bar{g}(\bar{g}^{-1}g)^{\frac{1}{2}}$$

- especially for $g = \bar{g}\mathrm{e}^{\bar{g}^{-1}h}$: $\quad \mathcal{B} = \bar{g}\mathrm{e}^{\frac{1}{2}\bar{g}^{-1}h}$
  [Eichhorn '13, Nink '14]

global spin base on the 2-sphere

- spherical coordinates (ϑ, φ)
- metric $(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \vartheta \end{pmatrix}$
- zweibein $({e_\mu}^a) = \begin{pmatrix} 1 & 0 \\ 0 & \sin \vartheta \end{pmatrix}$



- zweibein spin-connection: $\omega_1{}^{12} = 0, \quad \omega_2{}^{12} = -\cos\vartheta$
- flat Dirac matrices: $({\gamma_{(\mathrm{f})}}_a) = \begin{pmatrix} \sigma_1 \\ -\sigma_2 \end{pmatrix}$
- eigenfunctions $\psi^{(s)}_{\pm,n,l}$ of the Dirac operator read
  [Camporesi and Higuchi '96]

$$\psi^{(-)}_{\pm,n,l}(\vartheta,\varphi) = \frac{c_2(n,l)}{\sqrt{4\pi}} \mathrm{e}^{-\mathrm{i}(l+\frac{1}{2})\varphi} \begin{pmatrix} \Phi_{n,l}(\vartheta) \\ \pm\mathrm{i}(-1)^{n-l}\Phi_{n,l}(\pi-\vartheta) \end{pmatrix}$$

$$\psi^{(+)}_{\pm,n,l}(\vartheta,\varphi) = \frac{c_2(n,l)}{\sqrt{4\pi}} \mathrm{e}^{\mathrm{i}(l+\frac{1}{2})\varphi} \begin{pmatrix} \mathrm{i}(-1)^{n-l}\Phi_{n,l}(\pi-\vartheta) \\ \pm\Phi_{n,l}(\vartheta) \end{pmatrix}$$

the eigenfunctions have *strange* properties

$$\psi^{(s)}_{\pm,n,l}(\vartheta,\varphi+2\pi) = -\psi^{(s)}_{\pm,n,l}(\vartheta,\varphi)$$

$$\psi^{(s)}_{\pm,n,l=0}(\vartheta=0,\varphi) \sim \mathrm{e}^{s\mathrm{i}\frac{\varphi}{2}} \begin{pmatrix} 1-s \\ \pm(1+s) \end{pmatrix}$$

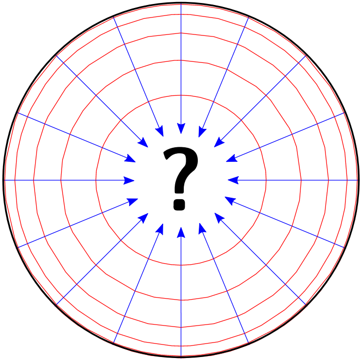
$$\psi^{(s)}_{\pm,n,l=0}(\vartheta=\pi,\varphi) \sim \mathrm{e}^{s\mathrm{i}\frac{\varphi}{2}} \begin{pmatrix} 1+s \\ \pm(1-s) \end{pmatrix}$$

$\Rightarrow$ **Why is that?**

- spherical coordinates are ill defined at the poles!
- need at least two patches in position space to cover the whole 2-sphere
- perform coordinate transformation

  $\Rightarrow$ zweibein is ill defined at the poles (hairy ball theorem)

  $\Rightarrow$ zweibein spin connection is singular at the poles



- cure problems by introducing globally well defined spin base

  $\gamma_\mu = \mathcal{S} e_\mu{}^a \gamma_{(\mathsf{f})a} \mathcal{S}^{-1}$, with $\mathcal{S} = \mathrm{e}^{-\mathrm{i}\frac{\varphi}{2}\sigma_3} \mathrm{e}^{-\mathrm{i}\frac{\vartheta-\pi}{2}\sigma_1}$

- circumvent hairy ball theorem by *clever* distribution of zeros of components $(\gamma_\mu)^I{}_J$

- spin connection $\hat{\Gamma}_\mu = \frac{\mathrm{i}}{2}\gamma_\mu$ is well defined

- globally well defined eigenfunctions of the Dirac operator

$$\hat{\psi}^{(s)}_{\pm,n,l} = \mathcal{S}\psi^{(s)}_{\pm,n,l}$$

$$\hat{\psi}^{(s)}_{\pm,n,l}(\vartheta, \varphi + 2\pi) = \hat{\psi}^{(s)}_{\pm,n,l}(\vartheta, \varphi)$$

$$\hat{\psi}^{(s)}_{\pm,n,l=0}(\vartheta = 0, \varphi) \sim \begin{pmatrix} \pm(1+s) \\ 1-s \end{pmatrix}$$

$$\hat{\psi}^{(s)}_{\pm,n,l=0}(\vartheta = \pi, \varphi) \sim \begin{pmatrix} 1+s \\ \pm(1-s) \end{pmatrix}$$

outlook

- existence of global spin bases on all metrizable manifolds?
- dynamics and effect of spin torsion $\Delta\Gamma_\mu$
- metric based path integral quantization of gravity in the presence of fermions

summary

- impose full nontrivial symmetry of Clifford algebra
  $\Rightarrow$ spinbase transformations: $\mathrm{SL}(d_\gamma, \mathbb{C})$
- spin metric h and canonical part of the spin connection $\hat{\Gamma}_\mu$ are determined by γ_μ
- common use of the vielbein as a function of the metric is justified from Clifford algebra perspective
- existence of a global spin base on the 2-sphere

Thank you for your attention!