Asymptotic safety in the sine–Gordon model[a]

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[a] J. Kovács, S.N., K. Sailer, arXiv:1408.2680, to appear in PRD

Outline

- Functional renormalization group method
- sine–Gordon model
  - Local potential approximation
  - Wave-function renormalization
- Massive sine–Gordon model
- Asymptotic safety in the sine–Gordon model
- The sine–Gordon model with an irrelevant coupling
- Duality

Motivations

sine–Gordon model

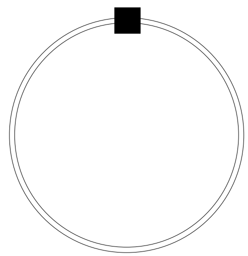
- bosonized version of the 2d fermionic Thirring model
- effective theory for low energy, low dimensional condensed matter systems
- same universality class as the 2d XY model and the Coulomb gas
- toy model for supersymmetry, string theory

Massive sine–Gordon model

- natural generalization of the sine–Gordon model
- bosonized version of the 2d quantum electrodynamics (massive Schwinger model)

Renormalization

- The functional RG method is a fundamental element of quantum field theory.
- The high energy (UV) action describes the small distance interaction between the elementary excitations. We look for the low energy IR (or large distance) behavior.
- The RG method gives a functional integro-differential equation for the effective action, which is called the *Wetterich equation*

$$\dot{\Gamma}_k = \frac{1}{2}\mathrm{Tr}\frac{\dot{R}_k}{R_k + \Gamma''_k} = \frac{1}{2}$$  ,

  where $' = \partial/\partial\varphi$, $^{\cdot} = \partial/\partial t$, and the symbol Tr denotes the momentum integral and the summation over the internal indices.
- The IR regulator has the form $R_k[\phi] = \frac{1}{2}\phi \cdot R_k \cdot \phi$. It is a momentum dependent mass like term, which serves as an IR cutoff. We use

$$R_k = p^2 \left(\frac{k^2}{p^2}\right)^b .$$

- The functional form of the effective action is assumed to be similar to the microscopic action

$$\Gamma_k \sim S_k .$$

Gradient expansion

The gradient expansion of the effective action is

$$\Gamma_k = \int d^d x \left[V_k(\phi) + \frac{1}{2} Z_k(\phi)(\partial_\mu \phi)^2 + H_1(\phi)(\partial_\mu \phi)^4 + H_2(\phi)(\Box \phi_x)^2 + \ldots \right].$$

The evolution equation for the potential is

$$\dot{V}_k(\phi) = \frac{1}{2} \int_p \frac{\dot{R}_k}{p^2 Z_k(\phi, p^2) + R_k + V_k''(\phi)}.$$

The momentum dependent wave-function renormalization evolves as

$$\begin{aligned} & q^2 \dot{Z}_k(\phi, q^2) \\ & \quad = \int_p \frac{\dot{R}_k \left[\frac{1}{2} q^2 Z_k'(\phi, q^2) + \frac{1}{2}(P{+}p)^2 Z_k'(\phi, (P{+}p)^2) + \frac{1}{2} p^2 Z_k'(\phi, p^2) + V_k'''(\phi) \right]^2}{[p^2 Z_k(\phi, p^2) + R_k + V_k''(\phi)]^2 [(P{+}p)^2 Z_k(\phi, (P{+}p)^2) + R_{k,P+p} + V_k''(\phi)]} \\ & \qquad - \int_p \frac{k\partial_k R_k [p^2 Z_k'(\phi, p^2) + V_k'''(\phi)]^2}{p^2 Z_k(\phi, p^2) + R_k + V_k''(\phi)} - \int_p \frac{k\partial_k R_k \frac{1}{2} q^2 Z_k''(\phi, q^2)}{[p^2 Z_k(\phi, p^2) + R_k + V_k''(\phi)]^2}. \end{aligned}$$

The 2d sine–Gordon model

Its effective action contains a sinusoidal potential of the form

$$\Gamma_k = \int \left[\frac{z}{2}(\partial_\mu \phi)^2 + u\cos\phi\right],$$

where z is the field independent wave-function renormalization and u is the coupling. The RG evolution equations for the couplings are

$$\begin{aligned}
\dot{u} &= \frac{1}{2}\mathcal{P}_1 \int_p \dot{R}G \\
\dot{z} &= \frac{1}{2}\mathcal{P}_0 \int_p \dot{R}\Big[-Z''G^2 + \left(\frac{2}{d}Z'^2p^2 + 4Z'V'''\right)G^3 \\
&\quad -2\left[V'''^2\left(\partial_{p^2}P + \frac{2}{d}p^2\partial^2 P\right) + \frac{4}{d}Z'p^2V'''\partial_{p^2}P\right]G^4 \\
&\quad +\frac{8}{d}p^2V'''^2\partial_{p^2}P^2G^5\Big]
\end{aligned}$$

with $G = 1/(zp^2 + R + V'')$, $P = zp^2 + R$
projections: $\mathcal{P}_1 = \int_\phi \cos(\phi)/\pi$ and $\mathcal{P}_0 = \int_\phi /2\pi$

The 2d sine–Gordon model

Symmetries

- Z_2
- periodicity

The conditions imply that the the effective (dimensionful) potential is zero.

What does the RG method say?

The linearized flow equation in LPA is ($\tilde{}$ denotes dimensionless quantities)

$$\dot{\tilde{u}} = \tilde{u}\left(-2 + \frac{1}{4\pi z}\right) + \mathcal{O}(\tilde{u}^2),$$

with any regulator. The equation can be solved analytically

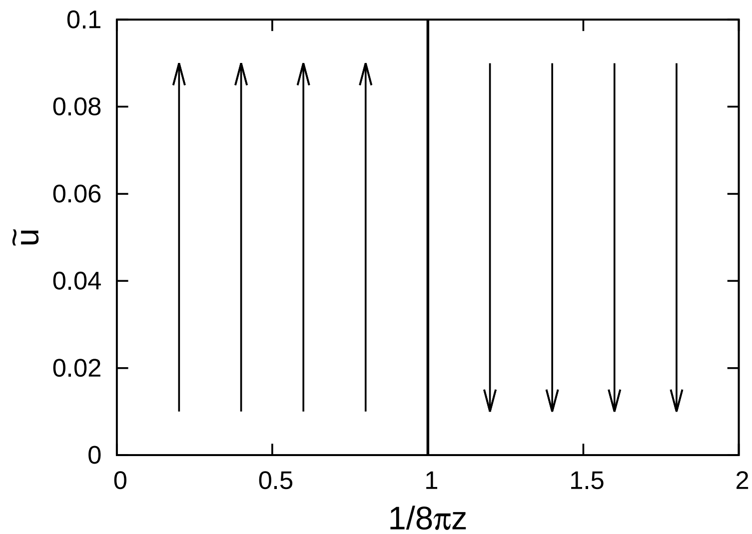
$$\tilde{u} = \tilde{u}(k_\Lambda)\left(\frac{k}{k_\Lambda}\right)^{\frac{1}{4\pi z}-2}.$$

The fixed point solution is $\tilde{u}^* = 0$ and z^* arbitrary.

The 2d sine–Gordon model

The SG model has **two phases**:

- $\frac{1}{z} > 8\pi \leftrightarrow$ **symmetric phase**. The coupling $\tilde{u}$ is irrelevant, the SG model is perturbatively nonrenormalizable
- $\frac{1}{z} < 8\pi \leftrightarrow$ (spontaneously) **broken** (symmetric) **phase**. The coupling $\tilde{u}$ is relevant, the SG model is perturbatively renormalizable



How one can distinguish the phases in the model?

⇒ The dimensionful coupling $\tilde{u}$ tends to zero, but the dimensionless one does not.

This idea can be generalized when we take into account the upper harmonics:

- **symmetric phase:**

$$\tilde{V}_{k\to 0}(\phi) = 0$$

- **broken phase:**

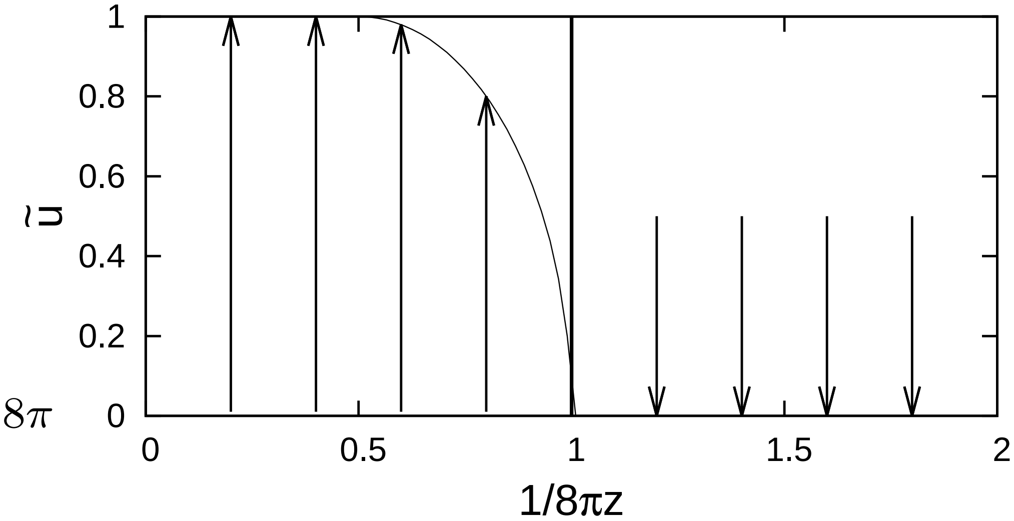
$$\tilde{V}_{k\to 0}(\phi) = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(n\phi)}{n^2} = -\frac{1}{2}\phi^2, \quad \phi \in [-\pi, \pi]$$

a concave function, which is repeated periodically in the field variable.

Local potential approximation

The 'exact' evolution equation is

$$\dot{\tilde{u}} = -2\tilde{u} + \frac{1}{2\pi\tilde{u}z}\left[1-\sqrt{1-\tilde{u}^2}\right],$$



with a constant z. The fixed points are

$$\begin{aligned}
\tilde{u}^* &= 1, \ \text{when} \ 0 < \frac{1}{z} < 4\pi \\
\tilde{u}^{*2} &= \frac{1}{2\pi z}\left(1 - \frac{1}{8\pi z}\right) \ \text{when} \ 4\pi < \frac{1}{z} < 8\pi \\
\tilde{u}^* &= 0 \ \text{when} \ \frac{1}{z} > 8\pi
\end{aligned}$$

Coleman point: $\tilde{u}^* = 0$ and $z_c^* = \frac{1}{8\pi}$

- in the symmetric phase the irrelevant scaling makes the model perturbatively nonrenormalizable
- in the broken phase we have finite IR values for the coupling $\tilde{u}$

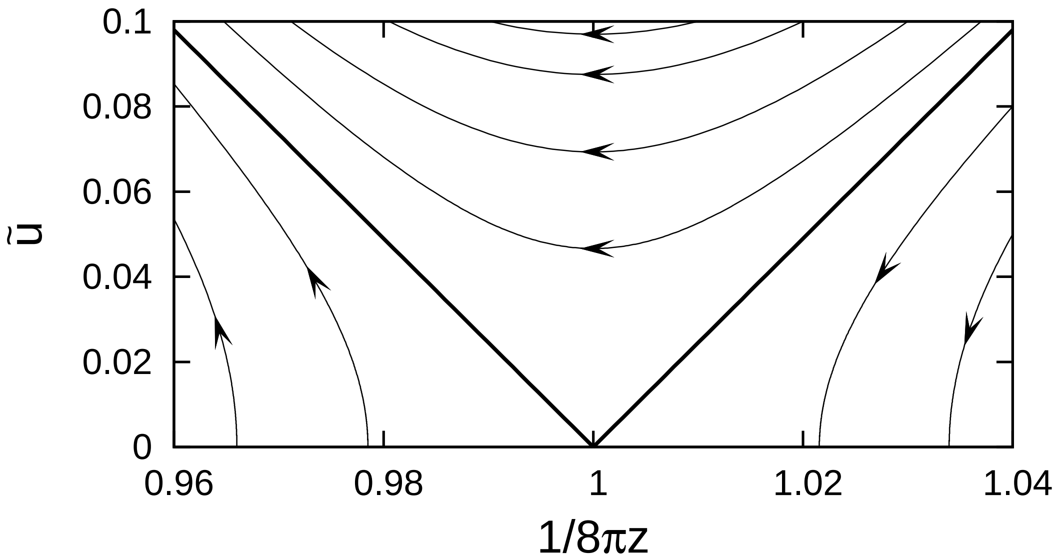
Wave-function renormalization

The linearized RG equations are

$$\dot{\tilde{u}} = -2\tilde{u} + \frac{1}{4\pi z}\tilde{u},$$

$$\dot{z} = -\frac{\tilde{u}^2}{z^{2-2/b}} c_b,$$

with $c_b = \frac{b}{48\pi}\Gamma\left(3 - \frac{2}{b}\right)\Gamma\left(1 + \frac{1}{b}\right)$.



The RG trajectories are hyperbolas

$$\tilde{u}^2 = \frac{2}{(8\pi)^{1-2/b} c_b}\left(z - \frac{1}{8\pi}\right)^2 + \tilde{u}^{*2},$$

The correlation length ξ is identified as $k_c \sim 1/\xi$ (singularity points). One obtains

$$\log \xi \approx \frac{\sqrt{\pi}}{8\sqrt{c_b}} \frac{1}{\tilde{u}^*} + \mathcal{O}(\tilde{u}^*), \quad \text{furthermore} \quad \tilde{u}^{*2} = kt + \mathcal{O}(t^2)$$

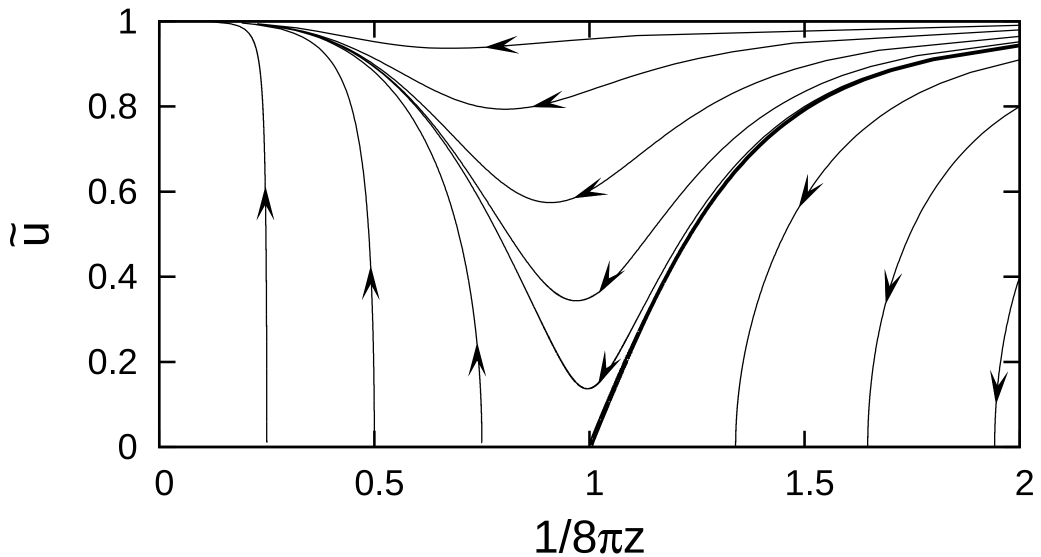
where the reduced temperature is $t \sim z(\Lambda) - z_s(\Lambda)$ ($z_s(\Lambda)$ is a point of the separatrix). We get

$$\boxed{\log \xi \propto t^{-\nu}} \quad \text{with} \quad \nu = \frac{1}{2}$$ **KT type phase transition**

Wave-function renormalization

The exact RG equations are $(b = 1)$

$$
\begin{aligned}
(2 + k\partial_k)\tilde{u} &= \frac{1}{2\pi\tilde{u}z}\left[1 - \sqrt{1 - \tilde{u}^2}\right] \\
k\partial_k z &= -\frac{1}{24\pi}\frac{\tilde{u}^2}{(1 - \tilde{u}^2)^{3/2}}
\end{aligned}
$$



There are seemingly no fixed points.

- Taylor expanding in $\tilde{u}$ we get $\tilde{u}^* = 0$, z (line of fixed points).
  - $1/z < 8\pi$ UV attractive
  - $1/z > 8\pi$ IR attractive
- Rescaling equations with ($\omega = \sqrt{1 - \tilde{u}^2}$, $\chi = 1/z\omega$ and $\partial_\tau = \omega^2 k\partial_k$)

$$
\begin{aligned}
\partial_\tau\omega &= 2\omega(1 - \omega^2) - \frac{\omega^2\chi}{2\pi}(1 - \omega), \\
\partial_\tau\chi &= \chi^2\frac{1 - \omega^2}{24\pi} - 2\chi(1 - \omega^2) + \frac{\omega\chi^2}{2\pi}(1 - \omega).
\end{aligned}
$$

We got an IR attractive fixed point at $\tilde{u}^* = 1$, $1/z^* = 0$.

Scheme dependence, IR divergences

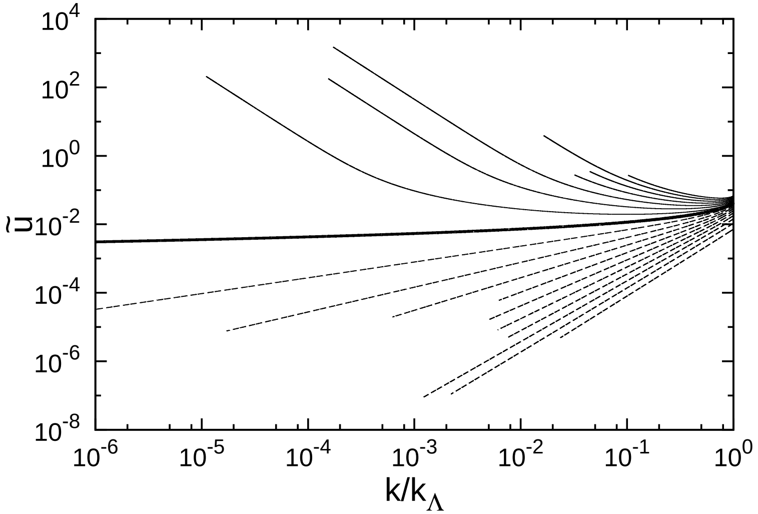
- we introduce $\bar{k} = \min(zp^2 + R)$
- for the power law IR regulator $R = p^2(k^2/p^2)^b$, with $b \geq 1$ we can calculate $\bar{k}$ analytically
- the corresponding renormalization scale is

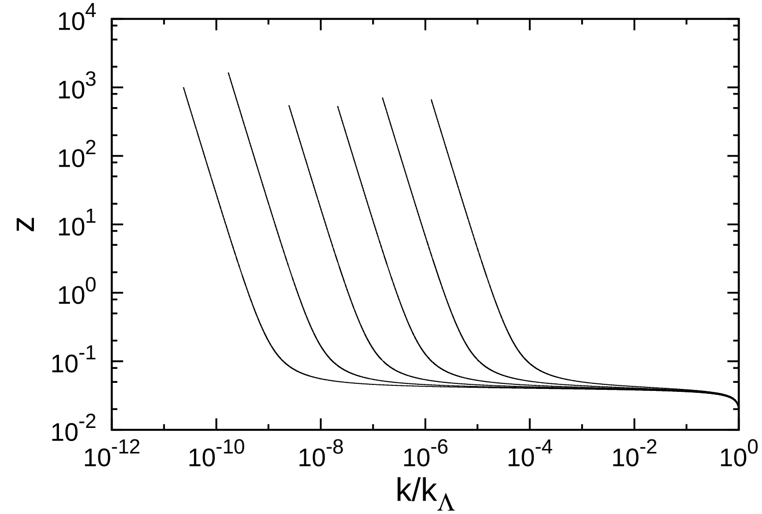
$$\bar{k}^2 = bk^2 \left(\frac{z}{b-1}\right)^{1-1/b}$$

- when $b = 1$, then $\bar{k} = k$
- we can remove the dimension of the coupling u by k or by $\bar{k}$

$$\tilde{u} = \frac{u}{k^2} \text{ and } \bar{u} = \frac{u}{\bar{k}^2}.$$

Scheme dependence, flow of the couplings

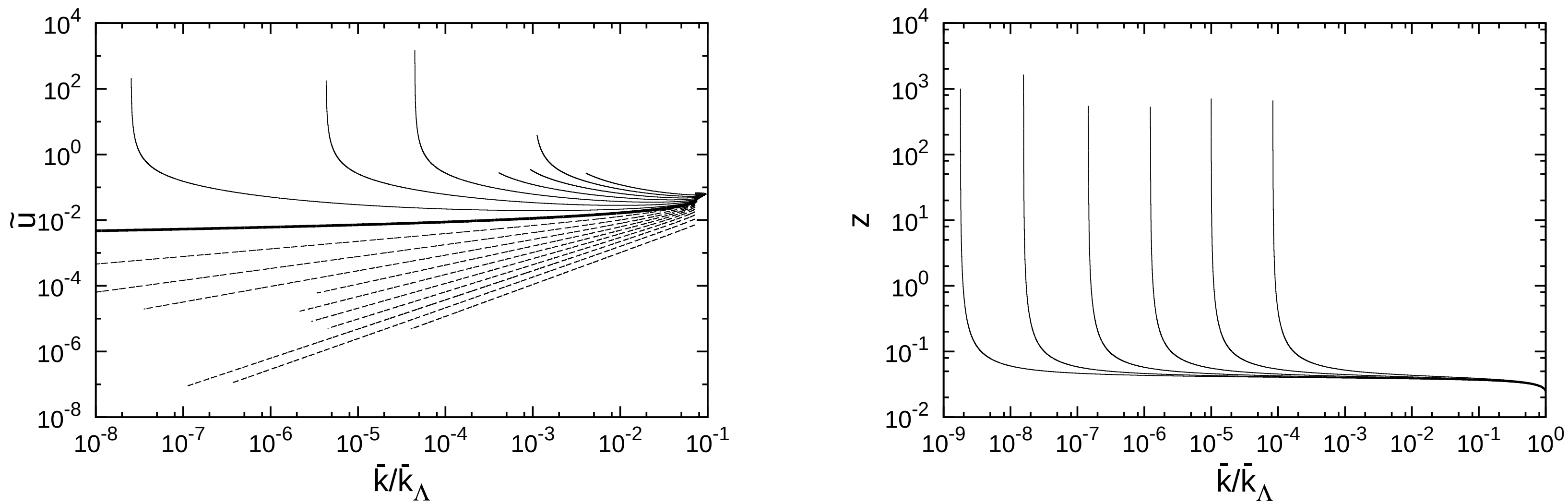




- $b = 2$
- the dashed (solid) lines represent the trajectories belonging to the (broken) symmetric phase, respectively, the wide line denotes the separatrix between the phases
- the couplings $\tilde{u}$ and z scales according to $k^{-\alpha}$ in the IR region (IR scaling regime exists)
- **symmetric phase**
  - the coupling $\tilde{u}$ tends to zero (α is negative and b dependent)
  - z is constant (not plotted) $\rightarrow$ LPA is a good approximation
- **broken phase**
  - the coupling $\tilde{u}$ diverges (α is positive and b dependent)
  - z also diverges

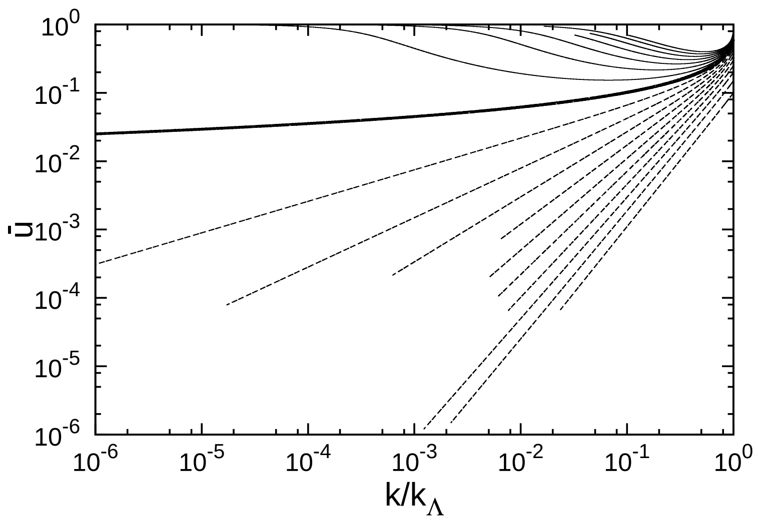
Scheme dependence, IR divergences

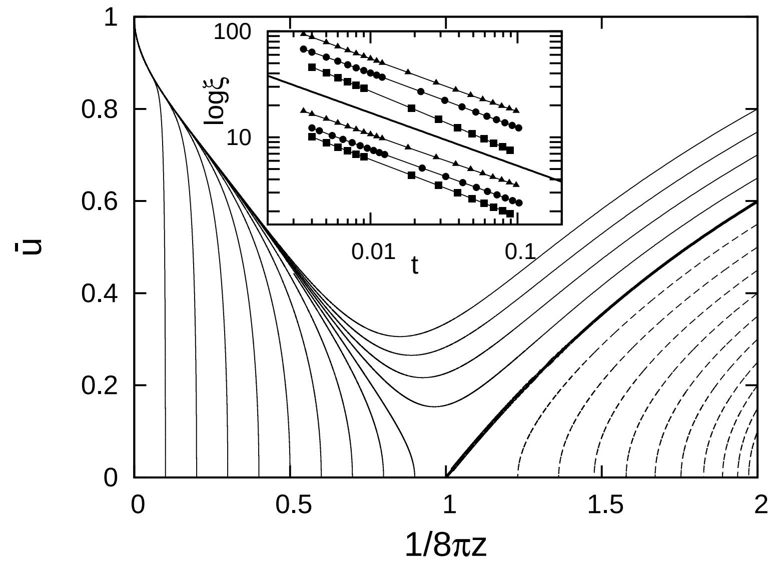
The flow of the couplings, singularities



- $b = 2$
- we changed the renormalization scale k to $\bar{k}$
- the scaling in the symmetric phase does not change
- the couplings $\tilde{u}$ and z blows up in the broken phase
- when $b \to 1$ then $\alpha \to \infty$, so we have a singular behavior

Scheme dependence, phase space





- $b = 5$
- the dimensionless coupling is $\bar{u} = u/\bar{k}^2$
- the inset shows the scaling of ξ w.r.t. the reduced temperature t
- the lower (upper) set of lines corresponds to the IR (KT) fixed point
- the triangle, circle and square correspond to $b = 2, 5, 10$, respectively
- in the middle a straight line with the slope $-1/2$ is drawn to guide the eye

Massive sine-Gordon model

The potential has the form

$$V = \frac{1}{2} m^2 \phi^2 + u \cos \phi.$$

The MSG model has no periodicity.
Under the mass scale the coupling scales as $\tilde{u} \sim k^{-2}$ independently on the initial conditions. It implies that in LPA the effective potential has the same form.
Then how can we distinguish the phases?
We use the **sensitivity matrix** which is defined as the derivatives of the running coupling constants with respect to the bare one

$$S_{n,m} = \frac{\partial \tilde{g}_n(k)}{\partial \tilde{g}_m(k_\Lambda)}.$$

It develops singularities when the UV and IR cutoffs are removed at the phase boundaries.

- **symmetric phase**: $S_{1,1} \sim k^{-2} \to \infty$
- **broken phase**: $S_{1,1} = 0$, since the RG evolution results in a universal effective potential in the IR limit.

The MSG model, evolution equations

The evolution of the mass decouples from $\tilde{u}$. The RG equations are

$$\begin{aligned}
\dot{\tilde{u}} &= -2\tilde{u} + \frac{1}{2\pi\tilde{u}z}\left[1+\tilde{m}^2 - \sqrt{(1+\tilde{m}^2)^2 - \tilde{u}^2}\right], \\
\dot{z} &= -\frac{1}{24\pi}\frac{\tilde{u}^2}{((1+\tilde{m}^2)^2-\tilde{u}^2)^{3/2}}, \\
\dot{\tilde{m}}^2 &= -2\tilde{m}^2.
\end{aligned}$$

The last equation gives

$$\tilde{m}^2 \sim k^{-2},$$

so the mass is a relevant coupling, furthermore we have no fixed points in the MSG model.

IR limit the MSG model exhibits a second order phase transition:

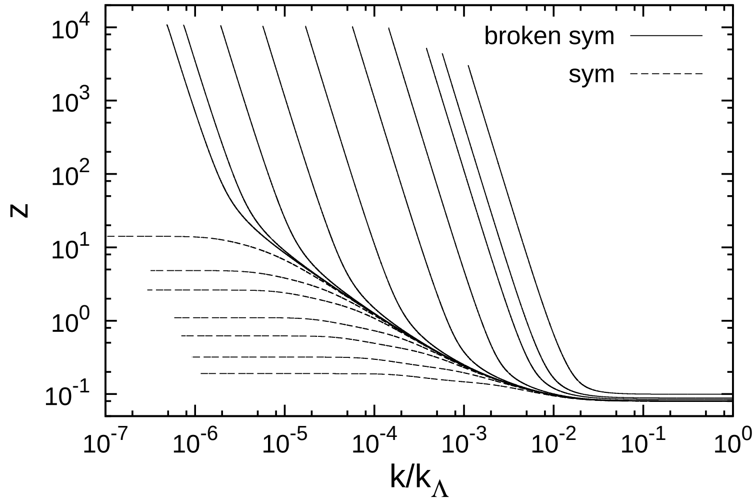
$$\xi \sim t^{-\nu}$$

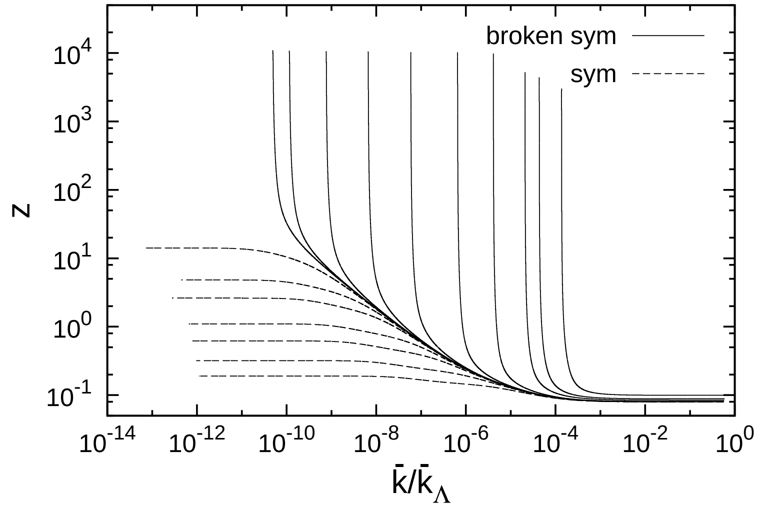
UV limit the mass can be neglected, so the model behaves as the SG model with an infinite order phase transition

$$\log \xi \sim t^{-\nu}$$

Wave-function renormalization

The evolution of z

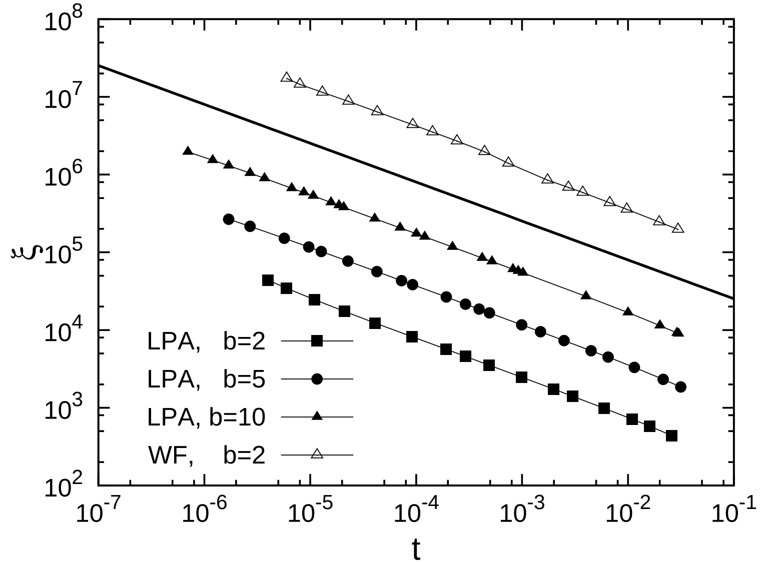




There is an IR scaling region of the MSG model, which exhibits a second order phase transition

$$\xi \propto t^{-\nu}.$$

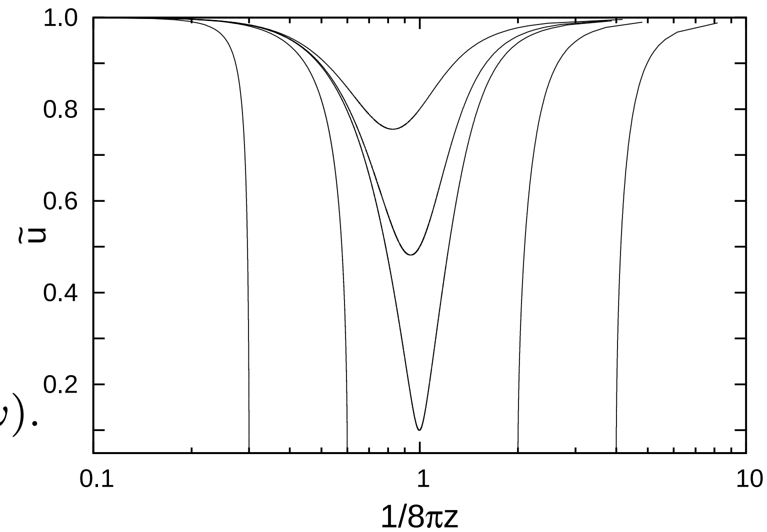
We numerically obtained that $\nu = \frac{1}{2}$.



Asymptotic safety

Rescaling:
$\omega = \sqrt{1-\tilde{u}^2}$, $\zeta = z\omega$ and $\partial_\tau = z\omega^2 k\partial_k$.

$$\partial_\tau \omega = 2\zeta\omega(1-\omega^2) - \frac{\omega^2}{2\pi}(1-\omega),$$
$$\partial_\tau \zeta = \left(2\zeta^2 - \frac{\zeta}{24\pi}\right)(1-\omega^2) - \frac{\omega\zeta}{2\pi}(1-\omega).$$



New fixed point can found at $z \to 0$ and $\tilde{u} \to 1$. The fixed point is UV attractive.

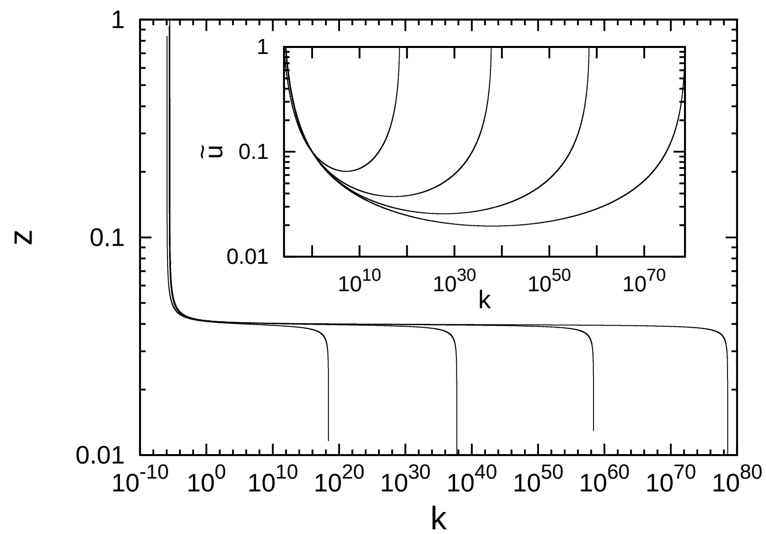
The fixed point of the 2d sine–Gordon model

- $\tilde{u}^* = 0$, z (line of fixed points)
  - $1/z < 8\pi$ UV attractive **GFP**
  - $1/z > 8\pi$ IR attractive
  - $1/z = 8\pi$ Coleman point
- $\tilde{u}^* = 1$, $1/z^* = 0$ IR attractive
- $\tilde{u}^* = 1$, $z^* = 0$ UV attractive **NGFP**

The model shows **asymptotic freedom** and **asymptotic safety**.

Asymptotic safety

- both in the IR and in the UV limits we get $\tilde{u} \to 1$.
- when $k \to 1$ then $z \to \infty$
- when $k \to \infty$ then $z \to 0$. The kinetic term tends to zero. Similar appears in the confining mechanism.



- The singularities shows up the limitation of the applicability of the models. New degrees of freedom appear.

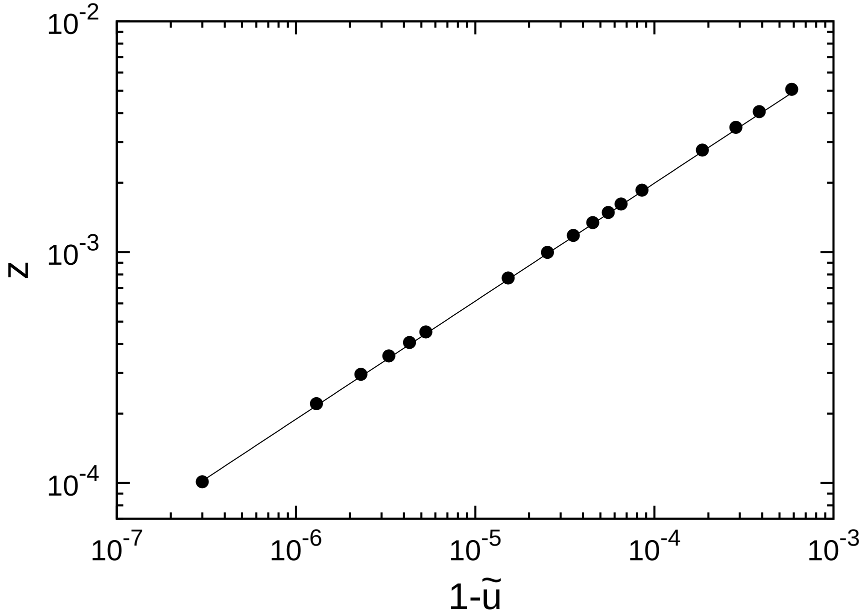
  IR: low energy limit, condensate (classical physics ?)

  UV: high energy limit, instead of vortices we have single spins
- around the UV NGFP we can also identify $\xi = 1/k_c$ and we get

$$\log \xi \propto t^{-\nu} \quad \nu = \frac{1}{2}.$$

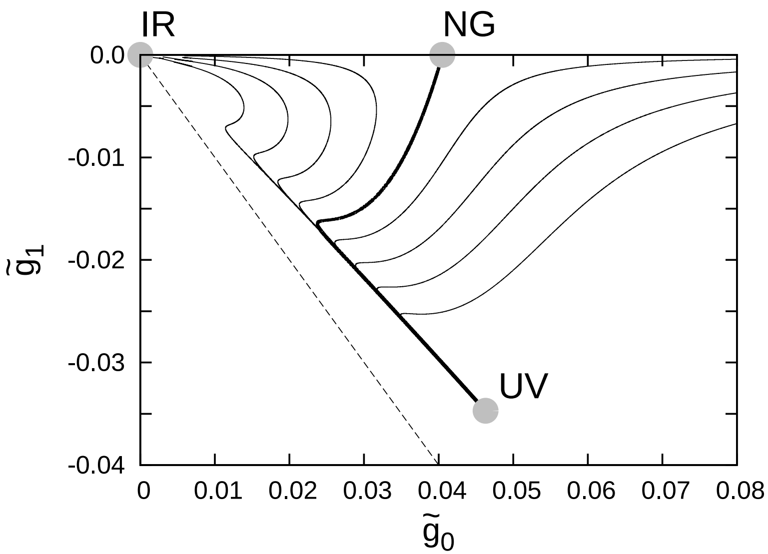
KT type phase transition. It originates from the Coleman point.

Asymptotic safety

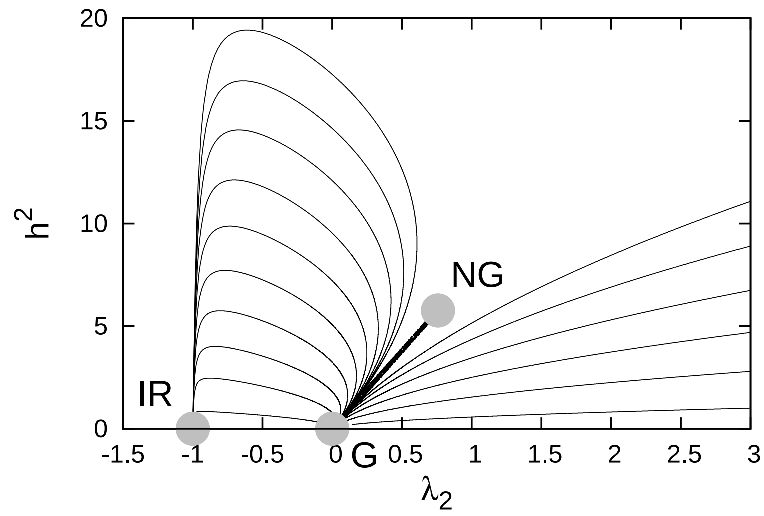


- The phase space does not show singularity.
- The sudden increase of $\tilde{u}$ and the sudden decrease of z compensate each other giving regular flows.
- around the UV NGFP we have $z = (1 - \tilde{u})^{3/2}$

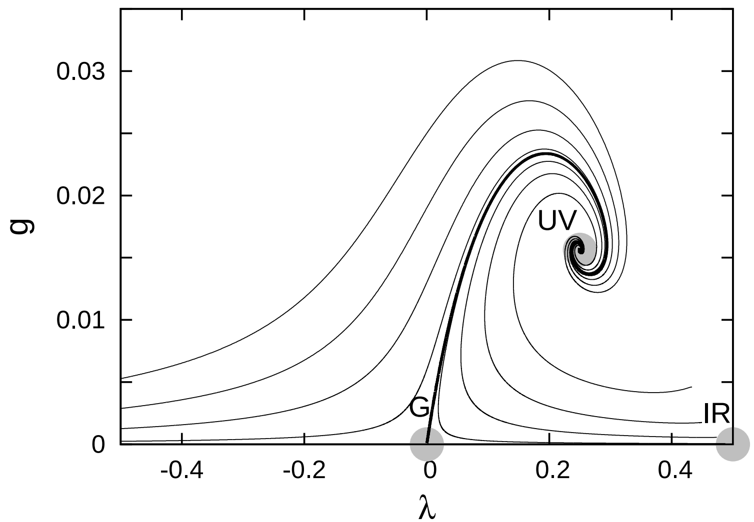
Asymptotically safe models



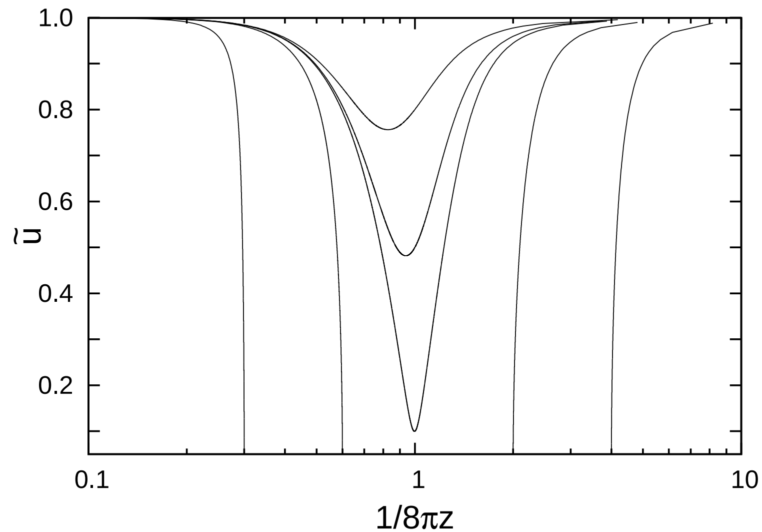
3d nonlinear σ model



3d Gross-Neveu model



4d Quantum Einstein Gravity



2d sine–Gordon model

The sine-Gordon model with an irrelevant coupling

The ZSG model is:

$$\begin{aligned}
\dot{\tilde{u}} &= -2\tilde{u} - \frac{1}{\tilde{u}}\int_y \left[1 - \frac{\tilde{Z}y+1}{[(\tilde{Z}y+1)^2-\tilde{u}^2]^{1/2}}\right] \\
\dot{z} &= \frac{\tilde{u}^2}{4}\int_y \left[\frac{-(2\partial_y\tilde{Z}+4\tilde{z}_1 y)(\tilde{Z}y+1)}{[(\tilde{Z}y+1)^2-\tilde{u}^2]^{5/2}} + \frac{y(\partial_y\tilde{Z})^2(4(\tilde{Z}y+1)^2+\tilde{u}^2)}{[(\tilde{Z}y+1)^2-\tilde{u}^2]^{7/2}}\right] \\
\dot{\tilde{z}}_1 &= 2\tilde{z}_1 + \frac{1}{48}\int_y \left[\frac{-24\tilde{z}_1(\tilde{Z}y+1)}{[(\tilde{Z}y+1)^2-\tilde{u}^2]^{5/2}}\right. \\
&\quad + \frac{(72\tilde{z}_1(\partial_y\tilde{Z})y+6(\partial_y\tilde{Z})^2+36\tilde{z}_1^2y^2)(4(1+zy+\tilde{z}_1y^2)^2+\tilde{u}^2)}{[(\tilde{Z}y+1)^2-\tilde{u}^2]^{7/2}} \\
&\quad + \frac{(-36(\partial_y\tilde{Z})^3y-108z_1(\partial_y\tilde{Z})^2y^2)(\tilde{Z}y+1)(4(\tilde{Z}y+1)^2+3\tilde{u}^2)}{[(\tilde{Z}y+1)^2-\tilde{u}^2]^{9/2}} \\
&\quad \left. + \frac{(18(\partial_y\tilde{Z})^4y^2)(8(\tilde{Z}y+1)^4+12(\tilde{Z}y+1)^2\tilde{u}^2+\tilde{u}^4)}{[(\tilde{Z}y+1)^2-\tilde{u}^2]^{11/2}}\right],
\end{aligned}$$

with $\tilde{Z} = zy + \tilde{z}_1 y^2$. We numerically obtained that

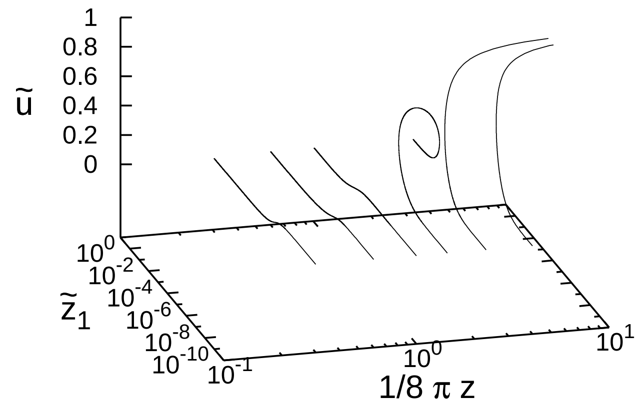
$$\tilde{z}_1 \sim k^2,$$

so it scales in an **irrelevant** manner.

- The RG equations become singular if the denominator

$$(\tilde{Z}y + 1)^2 - \tilde{u}^2 = 0.$$

  - when $\tilde{u} \to 1$ we have a singularity
  - when $\tilde{z}_1$ grows up faster than $\tilde{u}$ then there is no singularity.



UV limit: it shows a second order (probably Ising type) phase transition

- we have two phases in the UV.
- the correlation length scales as

$$\xi \sim t^{-\nu}, \quad \text{with} \quad \nu = \frac{1}{4}.$$

IR limit: there is an infinite order phase transition (from the SG model)

$$\log \xi \sim t^{-\nu}, \quad \text{with} \quad \nu = \frac{1}{2}.$$

Duality

The UV and the IR limits of the SG model seems self dual if we use the transformations

$$
\begin{aligned}
k &\leftrightarrow \frac{1}{k} \\
z &\leftrightarrow \frac{1}{z}.
\end{aligned}
$$

The duality can be extended to the ZSG and to the MSG models, if

$$\tilde{z}_1 \leftrightarrow \tilde{m}^2.$$

The ZSG and the MSG models become a dual pair.

model	**UV**	**IR**
SG	KT type, $\nu = 1/2$	KT type, $\nu = 1/2$
MSG	KT type, $\nu = 1/2$	Ising type, $\nu = 1/2$
ZSG	Ising type, $\nu = 1/4$	KT type, $\nu = 1/2$

Summary of the SG-type models and their fixed points.

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Thank You for Your attention