Quantum gravity with torsion and non-metricity

Carlo Pagani

work in collaboration with R. Percacci

arxiv: 1506.02882.

Asymptotic Safety seminar, 29/06/2015.

JOHANNES GUTENBERG UNIVERSITÄT MAINZ

Outline

- Motivations.
- Definition of torsion and non-metricity. Ansatz.
- RG flow of torsion and non-metricity due to the gravitons.
- Adding matter (fermions).

Motivation: exploring theory spaces of gravity

Asymptotic Safety proposal. Quantum Gravity can be fully described as a quantum field theory. A non-trivial UV fixed-point with finitely many relevant directions is searched for.

Some further choices are available:

- symmetries; $\longrightarrow$
  - diffeomorphism (e.g.: metric theories of gravity);
  - foliation preserving diffeomorphism (Horava-Lifshitz);
  - ...
- field content.

Field content:

- Metric formulation of gravity: the metric $g_{\mu\nu}$ carries the degree of freedom. Many aspects have been studied:

  - higher curvatures;
  - functional truncations;
  - matter fields;
  - bimetric aspects.

- Tetrad formulation of gravity (Harst and Reuter '12; Donà and Percacci '12).

- Tetrad+spin connection formulation of gravity (Daum and Reuter '13; Harst and Reuter '14).

- Dilaton gravity and non-integrable Weyl theory (Percacci '12; Codello, d'Odorico, Pagani, Percacci '12; Henz, Pawlowski, Rodigast, Wetterich '13; Pagani, Percacci '13).

$$\Downarrow$$

General case:

A generic connection has both torsion and non-metricity (defined in the next slide). However we did not see any physical effect so far. Maybe the quantum properties of these objects can explain why it is so.

Generic connection: torsion and non-metricity

In the $GL(n)$ formalism the metric is written as

$$g_{\mu\nu} = \theta^a{}_\mu \theta^b{}_\nu \gamma_{ab} \,. \tag{1}$$

A generic connection $A_\mu{}^a{}_b$ in the tangent bundle has both:

- torsion:

$$T_\mu{}^a{}_\nu = \partial_\mu \theta^a_\nu - \partial_\nu \theta^a_\mu + A_\mu{}^a{}_b \, \theta^b_\nu - A_\nu{}^a{}_b \, \theta^b_\mu \tag{2}$$

- non-metricity:

$$-Q_{\mu ab} = \partial_\mu \gamma_{ab} - A_\mu{}^c{}_a \gamma_{cb} - A_\mu{}^c{}_b \gamma_{ac} \,. \tag{3}$$

Going to the metric formalism we can write:

$$A_\alpha{}^\beta{}_\gamma = \Gamma_\alpha{}^\beta{}_\gamma + \phi_\alpha{}^\beta{}_\gamma \tag{4}$$

where

$$\phi_\alpha{}^\beta{}_\gamma = \alpha_\alpha{}^\beta{}_\gamma + \beta_\alpha{}^\beta{}_\gamma \,. \tag{5}$$

Here $\alpha_{\alpha\beta\gamma}$ is symmetric in (α,γ) and $\beta_{\alpha\beta\gamma}$ is antisymmetric in (β,γ) . The relation with torsion and non-metricity is given by the following formulas:

$$T_\alpha{}^\beta{}_\gamma = A_\alpha{}^\beta{}_\gamma - A_\gamma{}^\beta{}_\alpha \,, \qquad Q_{\mu\alpha\beta} = -\nabla^A_\mu g_{\alpha\beta} \tag{6}$$

and

$$T_{\alpha\beta\gamma} = \beta_{\alpha\beta\gamma} - \beta_{\gamma\beta\alpha} \,, \qquad Q_{\alpha\beta\gamma} = \alpha_{\alpha\beta\gamma} + \alpha_{\alpha\gamma\beta} \,. \tag{7}$$

The curvatures of A and Γ are related as follows:

$$F_{\mu\nu}{}^{\alpha}{}_{\beta} = R_{\mu\nu}{}^{\alpha}{}_{\beta} + \nabla_{\mu}\phi_{\nu}{}^{\alpha}{}_{\beta} - \nabla_{\nu}\phi_{\mu}{}^{\alpha}{}_{\beta} + \phi_{\mu}{}^{\alpha}{}_{\gamma}\phi_{\nu}{}^{\gamma}{}_{\beta} - \phi_{\nu}{}^{\alpha}{}_{\gamma}\phi_{\mu}{}^{\gamma}{}_{\beta}. \tag{8}$$

and (up to surface terms)

$$\begin{aligned} F_{\mu\nu}{}^{\mu\nu} \quad = \quad & R + \beta^{\alpha}{}_{\alpha\beta}\beta_{\gamma}{}^{\beta\gamma} + \beta_{\alpha\beta\gamma}\beta^{\beta\alpha\gamma} - \alpha_{\alpha\beta\gamma}\alpha^{\alpha\gamma\beta} + \alpha^{\alpha}{}_{\alpha\beta}\alpha_{\gamma}{}^{\beta\gamma} \\ & + \alpha_{\alpha\beta\gamma}\beta^{\alpha\beta\gamma} + \alpha^{\alpha}{}_{\alpha\beta}\beta_{\gamma}{}^{\beta\gamma} - \alpha^{\alpha}{}_{\beta\alpha}\beta_{\gamma}{}^{\beta\gamma}\,. \end{aligned} \tag{9}$$

This can be rewritten as:

$$\begin{aligned} F_{\mu\nu}{}^{\mu\nu} \quad = \quad & R + \frac{1}{4}T_{\alpha\beta\gamma}T^{\alpha\beta\gamma} + \frac{1}{2}T_{\alpha\beta\gamma}T^{\alpha\gamma\beta} + T_{\alpha}{}^{\alpha\beta}T_{\beta}{}^{\gamma}{}_{\gamma} \\ & + \frac{1}{4}Q_{\alpha\beta\gamma}Q^{\alpha\beta\gamma} - \frac{1}{2}Q_{\alpha\beta\gamma}Q^{\beta\alpha\gamma} - \frac{1}{4}Q_{\alpha\gamma}{}^{\gamma}Q^{\alpha\beta}{}_{\beta} + \frac{1}{2}Q_{\alpha}{}^{\alpha\beta}Q_{\beta\gamma}{}^{\gamma} \\ & - Q_{\alpha\beta\gamma}T^{\alpha\beta\gamma} + Q^{\alpha\beta}{}_{\beta}T_{\alpha}{}^{\gamma}{}_{\gamma} - Q_{\alpha}{}^{\alpha\beta}T_{\beta}{}^{\gamma}{}_{\gamma}\,. \end{aligned} \tag{10}$$

Ansatz

We consider the most general combination of terms appearing at first order in the curvature:

$$S = S_H + S_2 \tag{11}$$

where

$$S_H(g) = \bar{\kappa} \int d^4x \sqrt{g}\,(2\Lambda - R)\ , \qquad \bar{\kappa} = \frac{1}{16\pi G} \tag{12}$$

and

$$\begin{aligned} S_2 \quad &= \quad \int d^dx \sqrt{g} \Big[g_1 \beta_{\lambda\mu\nu} \beta^{\lambda\mu\nu} + g_2 \beta_{\lambda\mu\nu} \beta^{\mu\lambda\nu} + g_3 \beta^{\lambda}{}_{\lambda\mu} \beta^{\nu}{}_{\nu}{}^{\mu} \\ &+ g_4 \alpha_{\lambda\mu\nu} \alpha^{\lambda\mu\nu} + g_5 \alpha_{\lambda\mu\nu} \alpha^{\lambda\nu\mu} + g_6 \alpha^{\lambda}{}_{\lambda\mu} \alpha^{\mu\nu}{}_{n} + g_7 \alpha^{\lambda}{}_{\lambda\mu} \alpha^{\nu\mu}{}_{n} + g_8 \alpha^{\mu}{}_{\lambda\mu} \alpha_{\nu}{}^{\lambda\nu} \\ &+ g_9 \alpha_{\lambda\mu\nu} \beta^{\lambda\mu\nu} + g_{10} \alpha^{\lambda}{}_{\lambda\mu} \beta^{\nu}{}_{\nu}{}^{\mu} + g_{11} \alpha^{\lambda}{}_{\mu\lambda} \beta^{\nu}{}_{\nu}{}^{\mu} + \varepsilon - \text{terms} \Big]\,. \end{aligned} \tag{13}$$

The parity odd terms built via the epsilon tensor are the following:

$$
\begin{aligned}
\varepsilon-\text{terms} \quad = \quad & \varepsilon_{\alpha\beta\gamma\delta}\big(g_{12}\beta_{\rho}{}^{\alpha\beta}\beta^{\gamma\delta\rho}+g_{13}\beta^{\alpha\beta\rho}\beta^{\gamma\delta}{}_{\rho}+g_{14}\beta^{\rho\alpha\beta}\beta_{\rho}{}^{\gamma\delta} \\
& +g_{15}\beta^{\rho}{}_{\rho}{}^{\alpha}\beta^{\beta\gamma\delta}+g_{16}\alpha^{\rho\alpha\beta}\alpha_{\rho}{}^{\gamma\delta}+g_{17}\alpha^{\rho\alpha\beta}\beta_{\rho}{}^{\gamma\delta} \\
& +g_{18}\alpha^{\rho\alpha\beta}\beta^{\gamma\delta}{}_{\rho}+g_{19}\alpha^{\rho\alpha}_{\ \ \rho}\beta^{\beta\gamma\delta}+g_{20}\alpha^{\rho}{}_{\rho}{}^{\alpha}\beta^{\beta\gamma\delta}\big)\Big]\,.
\end{aligned}
\tag{14}
$$

The number of invariants can been reduced using the following identity:

$$
\varepsilon_{\alpha\beta[\gamma}{}^{[\eta}\delta_{\delta]}{}^{\theta]}=-\varepsilon_{\gamma\delta[\alpha}{}^{[\eta}\delta_{\beta]}{}^{\theta]}\,.
\tag{15}
$$

A basis of linearly independent monomials is:

$$
\begin{aligned}
\varepsilon-\text{terms} \quad = \quad & \varepsilon_{\alpha\beta\gamma\delta}\big(g_{13}\beta^{\alpha\beta\rho}\beta^{\gamma\delta}{}_{\rho}+g_{14}\beta^{\rho\alpha\beta}\beta_{\rho}{}^{\gamma\delta}+g_{16}\alpha^{\rho\alpha\beta}\alpha_{\rho}{}^{\gamma\delta} \\
& +g_{17}\alpha^{\rho\alpha\beta}\beta_{\rho}{}^{\gamma\delta}+g_{19}\alpha^{\rho\alpha}_{\ \ \rho}\beta^{\beta\gamma\delta}+g_{20}\alpha^{\rho}{}_{\rho}{}^{\alpha}\beta^{\beta\gamma\delta}\big)\Big]\,.
\end{aligned}
\tag{16}
$$

RG flow of torsion and non-metricity

We computed the 1-loop beta functions of the couplings $\{g_i\}$ via the flow equation for the EAA:

$$\partial_t \Gamma_k = \frac{1}{2} \mathrm{Tr} \left[\frac{\partial_t R_k}{\Gamma_k^{(2)} + R_k} \right] \tag{17}$$

$\alpha_{\alpha\beta\gamma}$ and $\beta_{\alpha\beta\gamma}$ do not have a kinetic term. Only gravitons enter in the loop. We expand the flow equation for small $\alpha_{\alpha\beta\gamma}$ and $\beta_{\alpha\beta\gamma}$:

$$\partial_t \Gamma_k = \frac{1}{2} \mathrm{Tr} \left[\frac{\partial_t R_k}{S_H^{(2)} + R_k} \right] - \frac{1}{2} \mathrm{Tr} \left[\frac{1}{S_H^{(2)} + R_k} S_2^{(2)} \frac{1}{S_H^{(2)} + R_k} \partial_t R_k \right] + \cdots . \tag{18}$$

For the Cosmological and the Newton constants one obtains beta functions of the following form:

$$\begin{aligned} \frac{d\tilde{\Lambda}}{dt} &= -2\tilde{\Lambda} + \frac{1}{2}A\tilde{G} - B\tilde{G}\tilde{\Lambda} \\ \frac{d\tilde{G}}{dt} &= (d-2)\tilde{G} - B\tilde{G}^2 \ . \end{aligned} \tag{19}$$

In the de-Donder gauge the r.h.s. of the flow equation for torsion and non-metricity couplings has the following form:

$$\partial_t \Gamma_k \quad \sim \quad -\frac{1}{\bar{\kappa}} Q_{d/2}\left(\frac{\partial_t R_k}{(P_k - 2\Lambda)^2}\right) \mathrm{Tr}\left(\mathbf{K}^{-1} S_2^{(2)}\right) \tag{20}$$

where

$$Q_n(f) = \tfrac{1}{\Gamma[n]} \int_0^\infty dz z^{n-1} f(z) \tag{21}$$

$$K^{\mu\nu}{}_{\rho\sigma} = \tfrac{1}{2}\left(\tfrac{\delta^\mu_\rho \delta^\nu_\sigma + \delta^\nu_\rho \delta^\mu_\sigma}{2} + \tfrac{1}{2} g^{\mu\nu} g_{\rho\sigma}\right) \ . \tag{22}$$

Computing the traces and using the relations among the various monomials one finds a set of beta functions of the following form:

$$\partial_t \tilde{g}_i = -(d-2)\,\tilde{g}_i + \kappa \sum_j c_{ij}\tilde{g}_j \tag{23}$$

where

$$\kappa = \frac{16\pi\tilde{G}}{(4\pi)^{d/2}} \frac{1}{\left(1-2\tilde{\Lambda}\right)^2} \frac{2}{(d/2)!}\,. \tag{24}$$

A convenient form is found by performing a linear transformation to the set of couplings $\{g_i\}$. This defines a new set of couplings $\{h_i\}$. For instance

$$h_4 = \tfrac{g_4+g_8}{3}\,. \tag{25}$$

These new couplings satisfy:

$$\partial_t \tilde{h}_j = (-(d-2)+\kappa\lambda_j)\tilde{h}_j\,. \tag{26}$$

The explicit form of the coefficients $\{\lambda_j\}$ is:

$$\begin{aligned}
\lambda_1 &= \frac{d^2-7d-12}{4}\,, \quad \lambda_2 = \frac{(d+1)(d-4)}{4}\,, \quad \lambda_3 = \frac{(d+1)(d-4)}{4}\,, \quad \lambda_4 = \frac{d^2-7d-16}{4}\,, \\
\lambda_5 &= \frac{(d+1)(d-4)}{4}\,, \quad \lambda_6 = \frac{(d+1)(d-4)}{4}\,, \quad \lambda_7 = \frac{(d+1)(d-4)}{4}\,, \\
\lambda_8 &= \frac{d^2-7d-4}{4}\,, \quad \lambda_9 = \frac{d^2-7d-16}{4}\,, \quad \lambda_{10} = \frac{(d+1)(d-4)}{4}\,, \quad \lambda_{11} = \frac{(d+1)(d-4)}{4}\,, \\
\lambda_{12} &= -(d+2)\,, \quad \lambda_{13} = -(d-1)\,, \quad \lambda_{14} = -d\,, \\
\lambda_{15} &= -d\,, \quad \lambda_{16} = -d + \frac{3}{2}\,, \quad \lambda_{17} = 0\,.
\end{aligned}$$

The explicit solution of the RG flow is:

$$G[k] = \frac{G_0}{1+\frac{1}{2}BG_0k^2} \tag{27}$$

and

$$h_j[k] = h_{j0}\left(1+\tfrac{1}{2}BG_0k^2\right)^{\lambda_j/B\pi}\;. \tag{28}$$

Our system of beta functions has the following fixed points:

- Gaussian fixed point for $\left(\tilde{\Lambda}, \tilde{G}\right)$ plus a Gaussian fixed point for the couplings $\left\{\tilde{h}_i\right\}$.

- UV non-Gaussian fixed point for $\left(\tilde{\Lambda}, \tilde{G}\right)$ plus a Gaussian fixed point for the couplings $\left\{\tilde{h}_i\right\}$ which can be seen from:

$$\partial_t \tilde{h}_j = (-(d-2) + \kappa\lambda_j)\tilde{h}_j \, . \tag{29}$$

- Gaussian fixed point for $\left(\tilde{\Lambda}, \tilde{G}\right)$ plus a Gaussian fixed point for the couplings $\left\{1/\tilde{h}_i\right\}$.

The Holst subsector

A particular combination of our ansatz corresponds to the Holst action:

$$S_{Holst} = \bar{\kappa} \int d^4x \left[\varepsilon_{abcd} \left(2\Lambda\, \theta^a \wedge \theta^b \wedge \theta^c \wedge \theta^d - F^{ab} \wedge \theta^c \wedge \theta^d \right) + \frac{1}{\gamma} F_{ab} \wedge \theta^a \wedge \theta^b \right]$$

We can consider the curvatures of the connections $\Gamma + \beta$ or $\Gamma + \alpha + \beta$.

The correspondence with our ansatz is given by:

$$-g_2 = g_3 = \bar{\kappa} \; ; \qquad g_{13} = \bar{\kappa}/\gamma \; ; \qquad g_i = 0, \; \text{ for } i \neq \{2, 3, 13\} \; . \tag{30}$$

The above relation is broken by the flow equation. The same goes for the connection $\Gamma + \alpha + \beta$.

Adding matter: fermions

Scalar and vector field actions which do not need a gravitational connection. We consider the interaction of (minimally coupled) fermions with torsion (they do not couple to non-metricity):

$$S_{1/2} = \frac{i}{2}\int dx\,\det(\theta)\left[\bar{\psi}\gamma^a\theta_a^\mu\hat{\nabla}_\mu\psi - \hat{\nabla}_\mu\bar{\psi}\gamma^a\theta_a^\mu\psi\right] \tag{31}$$

After integration by parts we get the following Dirac operator:

$$\mathcal{D} = \gamma^a\theta_a^\mu\hat{\nabla}_\mu + \frac{1}{2}T_\mu{}^\rho{}_\rho\bar{\psi}\gamma^a\theta_a^\mu \tag{32}$$

The square of the Dirac operator has the form:

$$\Delta = -\hat{\nabla}^2 + B^\rho\hat{\nabla}_\rho + X\,. \tag{33}$$

In our case:

$$
\begin{aligned}
B^{\rho} &= \frac{1}{4}\left[\gamma^{\mu},\gamma^{\nu}\right] T_{\mu}{}^{\rho}{}_{\nu} - T^{\rho\alpha}{}_{\alpha}\mathbb{I} \\
X &= \frac{1}{4}\hat{F}_{\mu\nu}{}^{\mu\nu}\mathbb{I} - \frac{1}{2}\hat{\nabla}_{\mu}T^{\mu\alpha}{}_{\alpha}\mathbb{I} - \frac{1}{4}\left[\gamma^{\mu},\gamma^{\nu}\right]\left(\hat{\nabla}_{\mu}T_{\nu}{}^{\alpha}{}_{\alpha}\right) - \frac{1}{4}T^{\mu\alpha}{}_{\alpha}T_{\mu}{}^{\beta}{}_{\beta}\mathbb{I}.
\end{aligned}
\tag{34}
$$

The traces can be computed using the heat kernel for torsionful connection.

In particular we have:

$$
\begin{aligned}
b_2\left(\hat{\Delta}\right) &= \frac{1}{(4\pi s)^{d/2}}\int\sqrt{g}s\mathbb{I}\left[\frac{R}{6} - X + \frac{1}{2}\nabla_{\mu}T^{\mu\alpha}{}_{\alpha} - \frac{1}{4}T^{\mu\alpha}{}_{\alpha}T_{\mu}{}^{\beta}{}_{\beta}\right. \\
&\left. -\frac{1}{2}T_{\alpha}{}^{\beta}{}_{\beta}B^{\alpha} + \frac{1}{2}\nabla_{\mu}B^{\mu} - \frac{1}{4}B_{\mu}B^{\mu}\right].
\end{aligned}
\tag{35}
$$

The contribution to the flow equation due to a fermion is:

$$
\begin{aligned}
\partial_t\Gamma_k &= -\frac{1}{2}\mathrm{tr}\left[Q_{d/2-1}\left(\frac{\partial_t R_k\left(\Delta\right)}{\Delta + R_k\left(\Delta\right)}\right)B_2\left(\Delta\right)\right] \\
&= -\frac{1}{(4\pi)^{d/2}}\frac{k^{d-2}}{\left(\frac{d}{2}-1\right)!}2^{\left[\frac{d}{2}\right]}\left[\frac{1}{4}\beta_{\alpha\beta\gamma}\beta^{\alpha\beta\gamma} - \frac{1}{2}\beta_{\alpha\beta\gamma}\beta^{\alpha\gamma\beta}\right].
\end{aligned}
\tag{36}
$$

The beta functions are modified as follows:

$$\begin{aligned} \partial_t \tilde{g}_1 &= -(d-2)\,\tilde{g}_1 + \kappa \frac{1}{4}((d-7)d-12)\tilde{g}_1 - \frac{1}{(4\pi)^{d/2}} \frac{1}{\left(\frac{d}{2}-1\right)!} 2^{\left[\frac{d}{2}\right]-2} \\ \partial_t \tilde{g}_2 &= -(d-2)\,\tilde{g}_2 + \kappa \frac{1}{4}(d-4)(d+1)\tilde{g}_2 + \frac{1}{(4\pi)^{d/2}} \frac{1}{\left(\frac{d}{2}-1\right)!} 2^{\left[\frac{d}{2}\right]-1} \end{aligned} \tag{37}$$

New fixed points appear:

	$\tilde{\Lambda}$	$\tilde{G}$	$\tilde{g}_1$	$\tilde{g}_2$	$\tilde{g}_3$
FP_1	0	0	-0.00316629	0.00633257	0
FP_2	-0.3	1.88496	-0.0018591	0.00633257	0

	$\tilde{\Lambda}$	$\tilde{G}$	$1/\tilde{g}_1$	$1/\tilde{g}_2$	$1/\tilde{g}_3$
FP_3	-0.3	1.88496	-537.893	157.914	0
FP_4	-0.3	1.88496	-537.893	0	0
FP_5	0	0	-315.827	157.914	0
FP_6	0	0	-315.827	0	0
FP_7	0	0	0	157.914	0
FP_8	-0.3	1.88496	0	157.914	0
FP_9	-0.3	1.88496	0	0	0
FP_{10}	0	0	0	0	0

Summary

- We considered the most general combination of gravitational terms with torsion and non-metricity up to the first order in the curvatures and computed the one-loop beta functions of the couplings.

- We confirm the Asymptotic Safety fixed point in this approximation.

- The Holst action is not preserved under renormalization.

- If fermions are added new fixed points appear in the torsion sector.

Thank you!!