A proper fixed functional for four-dimensional Quantum Einstein Gravity

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$f(R)$-gravity

renormalization group flow of $f_k(R)$

$$\Gamma_k^{\text{grav}}[g_{\mu\nu}] = \int \mathrm{d}^d x \, \sqrt{g} \; f_k(R)$$

- infinitely many couplings
- recent progress [Benedetti,Dietz, Eichhorn, Litim, Morris, Percacci, Vacca, ...]
- many interesting and non-trivial features in the functional setup
- background: four-sphere

beta functions

dimensionless quantities:

$$R =: k^2 r, \qquad f_k(R) =: k^d \varphi_k(R/k^2)$$

- flow equation: partial differential equation for $\varphi_k(r)$
- fixed functions: k stationary solutions $\iff \partial_t \varphi_k(r) = 0$

singular structure

fixed functions $\varphi(r)$ satisfy non-linear ODEs
in general there are singular points $r_{\mathrm{sing},i}$ in the ODE

BUT: $k \in [0,\infty)$ and $r = \frac{R}{k^2} \implies \varphi(r)$ should exist for $r \in [0,\infty)$

singularity count [Dietz,Morris]:

number of singular points - order of the ODE $= 0$

regularity at singular points reduces the number of free parameters

regular singular points and pole crossing

generic ODE: $y^{(n)}(x) = f(y^{(n-1)}, \dots, y', y, x)$

r.h.s. can have singular points

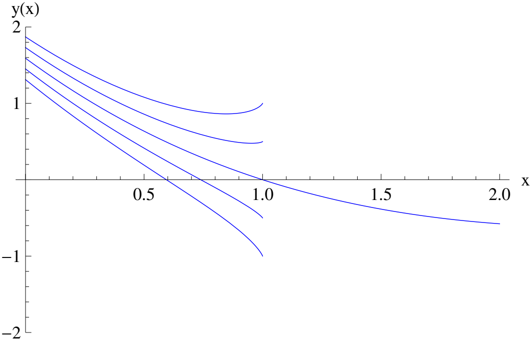
$$f(y^{(n-1)}, \dots, y', y, x) = \frac{e(y^{(n-1)}(x_0), \dots, y'(x_0), y(x_0), x_0)}{x - x_0} + \mathcal{O}\left((x - x_0)^0\right)$$

- pole crossing solution $\iff$ $e|_{x=x_0} = 0$ (regularity condition)
- additional boundary condition reduces number of free parameters

analytic example

$$y''(x) = -\frac{y(x)}{x-1}, \qquad y(0) = y_0,\ y'(0) = y_1$$

solutions are modified Bessel functions I_n



regularity condition $y(1) = 0$: $y_1 = -\left(1 + \frac{I_2(2)}{I_2(2)}\right) y_0 \approx -1.433 y_0$

flow equation for $f_k(R)$-gravity

AIM:

- admit globally well-defined solutions
- satisfy singularity count

geometric flow [DeWitt, Pawlowski]

field space admits fiber bundle structure $\mathfrak{F} \xrightarrow{\pi} \mathfrak{F}/_{\mathfrak{G}}$ with $\mathfrak{F} \equiv \mathrm{Riem}(M)$, $\mathfrak{G} \equiv \mathrm{Diff}(M)$

local trivialization:

$$g_{\mu\nu} \mapsto (h^A, \phi^\alpha)$$

h^A are coordinates in $\mathfrak{F}/_{\mathfrak{G}}$ and ϕ^α in $\mathfrak{G}$

path integral $\int_{\mathfrak{F}/_{\mathfrak{G}}} \mathrm{D}h^A$:

$$\partial_t \Gamma_k[h^A; \bar{g}] = \frac{1}{2} \, \mathrm{Tr} \left[\left(\Gamma_k^{(2)} + \mathcal{R}_k \right)^{-1} \partial_t \mathcal{R}_k \right]$$

with $\Gamma_k^{(2)AB}(x,y) \equiv \frac{1}{\sqrt{\bar{g}(x)}} \frac{1}{\sqrt{\bar{g}(y)}} \frac{\delta^2 \Gamma_k}{\delta h^A(x) \, \delta h^B(y)}$

TT-decomposition

gauge tranformation:

$$\delta_Q\, h_{\mu\nu} = \mathcal{L}_v h_{\mu\nu} = D_\mu\, v_\nu^{\mathrm{T}} + D_\nu\, v_\mu^{\mathrm{T}} + 2D_\mu D_\nu v$$

here $v_\mu = v_\mu^{\mathrm{T}} + D_\mu v$.

TT-decomposition:

$$h_{\mu\nu} = \tilde{h}_{\mu\nu}^{\mathrm{T}} + \bar{D}_\mu \tilde{\xi}_\nu + \bar{D}_\nu \tilde{\xi}_\mu + 2\bar{D}_\mu \bar{D}_\nu \tilde{\sigma} - \tfrac{1}{d}\, \bar{g}_{\mu\nu}\, \tilde{\chi}$$

here: $\tilde{\chi} = \tilde{h} - 2D^2\tilde{\sigma}$, $\bar{D}^\mu \tilde{h}_{\mu\nu}^{\mathrm{T}} = \bar{g}^{\mu\nu}\tilde{h}_{\mu\nu}^{\mathrm{T}} = 0$, $\bar{D}^\mu \tilde{\xi} = 0$ and $\tilde{h} = \bar{g}^{\mu\nu} h_{\mu\nu}$

gauge transformation:

$$\tilde{h}_{\mu\nu}^{\mathrm{T}} \mapsto \tilde{h}_{\mu\nu}^{\mathrm{T}}\,, \quad \tilde{\xi}_\mu \mapsto \tilde{\xi}_\mu + v_\mu^{\mathrm{T}}\,, \quad \tilde{\sigma} \mapsto \tilde{\sigma} + v\,, \quad \tilde{\chi} \mapsto \tilde{\chi}$$

$\Longrightarrow$ $\tilde{h}_{\mu\nu}^{\mathrm{T}}$ and $\tilde{\chi}$ are gauge invariant fields

Landau DeWitt gauge $\alpha \to 0$

gauge fixing condition [0705.1769,0712.0445,hep-th/9502109]

$$F_\mu = \bar{D}^\nu h_{\mu\nu} - \tfrac{1}{d}\bar{D}_\mu h$$

F_μ is independent of $\tilde{h}^{\mathrm{T}}_{\mu\nu}$ and $\tilde{\chi}$

and specific regulator choice:

$\Longrightarrow$ cancellation between gauge-degrees of freedom and ghost modes

$$\Longrightarrow \; \partial_t \Gamma_k[h^A; \bar{g}] = \tfrac{1}{2}\,\mathrm{Tr}\left[\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1} \partial_t \mathcal{R}_k\right] + \mathcal{O}(\alpha)$$

physical flow

flow equation

$$\partial_t \Gamma = \frac{1}{2}\,\mathrm{Tr}\left(\Gamma^{(2)}_{h^{\mathrm{T}}h^{\mathrm{T}}} + R_{h^{\mathrm{T}}h^{\mathrm{T}}}\right)^{-1}\partial_t R_{h^{\mathrm{T}}h^{\mathrm{T}}} + \frac{1}{2}\,\mathrm{Tr}\left(\Gamma^{(2)}_{\chi\chi} + R_{\chi\chi}\right)^{-1}\partial_t R_{\chi\chi}$$

no unphysical singularities originating form gauge-fixing and ghost traces!

coarse graining operator

$$\Box_{(2)} \equiv -\bar{D}^2 + \alpha R \qquad \Box_{(0)} \equiv -\bar{D}^2 + \beta R$$

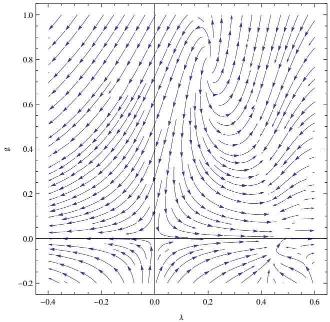
replacement: $\Box_{(s)} \mapsto \Box_{(s)} + R_k(\Box_{(s)})$

$\Longrightarrow$ third order ODE $\varphi^{(3)}(r) = \ldots$

benchmark

Einstein-Hilbert

- $f_k(R) = \frac{1}{16\pi G_k}\{2\Lambda_k - R\}$
- $\alpha = \beta = 0$
- NGFP: $g_* = 0.781$, $\lambda_* = 0.203$, $\tau_* = 0.16$
- $\theta' = 2.929$, $\theta'' = 2.965$



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polynomial truncations $f(R) = \sum_{n=0}^{N} g_n R^n$

- local heat-kernel expansion
- good agreement with previous works [0705.1769,0712.0445,0805.2909,1410.4815,...]
- minimization of $\theta_m(\alpha,\beta)$ favors $\alpha, \beta \neq 0$

operator trace for full $f(R)$

spectral sum (including non-analytic terms):

$$\mathrm{Tr}_{(s)}\, W_{(s)}(\Box_{(s)}) = \sum_i D_i^{(s)} W_{(s)}(\lambda_i^{(s)})$$

eigenvalues $\lambda_i^{(s)}$ and multiplicities $D_i^{(s)}$ of $\Box_{(s)}$

integrating out eigenmodes:

$$\begin{aligned} \text{optimised cutoff} \implies & W_{(s)}(\lambda_i) \propto \theta(k^2 - \lambda_i^{(s)}) \\ \implies & W_{(s)}(\lambda_i) \neq 0 \iff k^2 \geq \lambda_i^{(s)} \end{aligned}$$

- every time k crosses $\lambda_i^{(s)}$ the eigenmode is integrated out
- spectral sum is a finite sum ($\lambda_i \propto R$)

if lowest eigenvalue $\lambda_0^{(s)} > 0$: $\mathrm{Tr}_{(s)} W_{(s)}(\Box_{(s)}) = 0$ for $k^2 < k_{\mathrm{term}}^2 = \lambda_0^{(s)}$

for the TT and the scalar trace (R fixed):

$$r_{(2),\mathrm{term}} = \frac{3}{2+3\alpha}\,, \qquad r_{(0),\mathrm{term}} = \frac{1}{\beta}$$

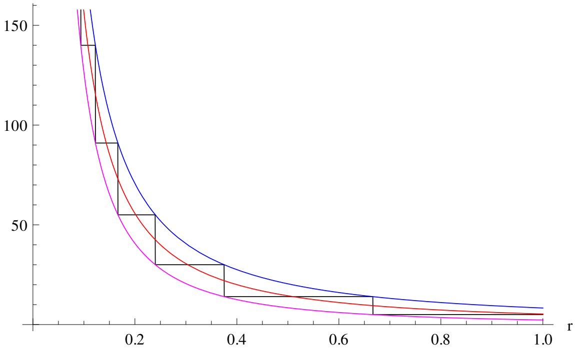
equal lowest eigenvalue (ELE):

$$k_{\mathrm{term}}^2 = \lambda_0^{(2)} = \lambda_0^{(0)} \iff \alpha = \beta - \frac{2}{3}$$

classical scaling $r^{d/2}$ for $r > r_{\mathrm{term}} \equiv r_{(2),\mathrm{term}} = r_{(0),\mathrm{term}}$

(here globally well-defined $r \in [0, r_{\mathrm{term}}]$)

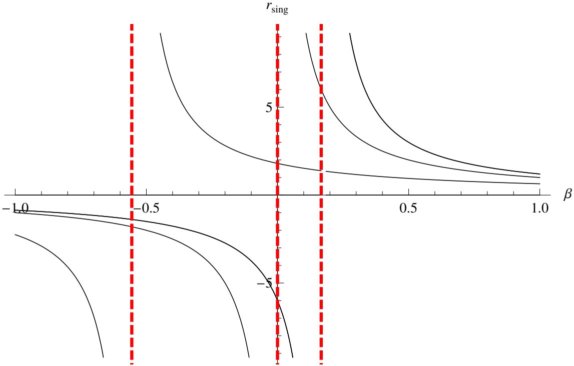
smoothing the step function [Benedetti,...]



globally well-defined: $r \in [0, \infty)$

singular structure

scalar trace: $r\, c_4(r;\beta)\, \varphi^{(3)}(r)$, poles: $c_4(r_{\text{sing}};\beta) = 0$



for $0 < \beta \leq \frac{1}{6}$ three fixed singularities

$\beta = \frac{1}{6}$: $r_{1,\text{sing}} = 0$, $r_{2,\text{sing}} = \frac{18}{13}$ and $r_{1,\text{sing}} = 6$

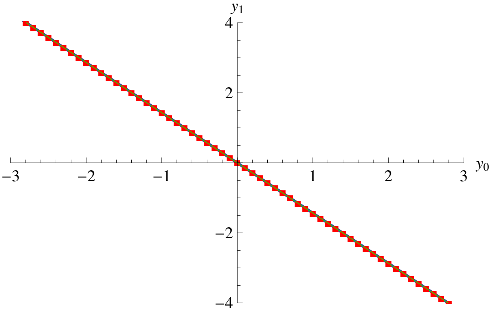
numerical analysis

shooting method: analytic example

IWP: $y''(x) = -\frac{y(x)}{x-1}$, $\qquad y(0) = y_0,\ y'(0) = y_1$

- discretize y_0,y_1
- numerically integrate: $x \in [0, x_{\text{sing}} - \varepsilon]$
- num. solutions:

  $(y_0, y_1) \mapsto e(x; y_0, y_1)|_{x=x_{\text{sing}}}$
- regular solutions $e \approx 0$



shooting algorithm yields $y_1 = -1.434 y_0 \implies$ very good agreement

regularity conditions

Laurent expanding r.h.s.

$$\varphi'''(r) = \frac{e_i(\varphi''(r_i), \varphi'(r_i), \varphi(r_i), r_i)}{r - r_i} + \text{regular terms}$$

three regularity (boundary) conditions: $e_3 = 0$, $e_2 = 0$ and $e_1 = 0$

for $n \geq 2$ Taylor coefficients are fixed

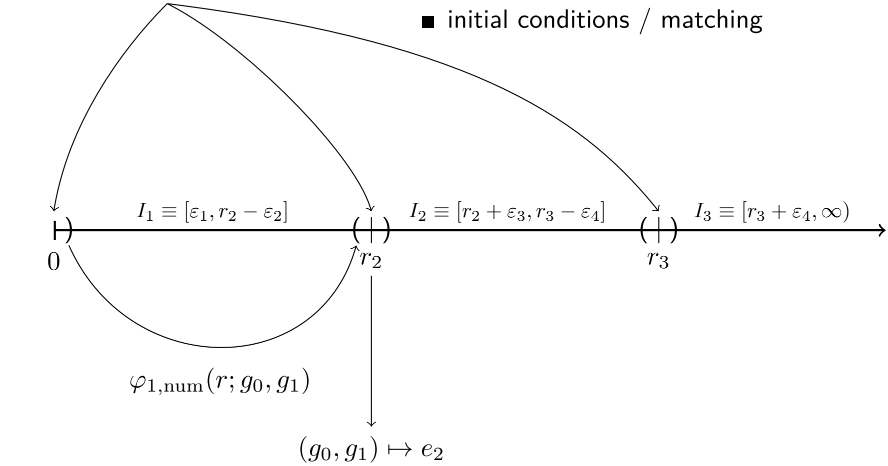
$$\varphi_1(r) = g_0 + g_1 r + \sum_{n \geq 2} g_n(g_0, g_1)\, r^n$$

$$\varphi_2(r) = b_0 + b_1(r - r_2) + \sum_{n \geq 2} b_n(b_0, b_1)\,(r - r_2)^n$$

$$\varphi_3(r) = c_0 + c_1(r - r_3) + \sum_{n \geq 2} c_n(c_0, c_1)\,(r - r_3)^n$$

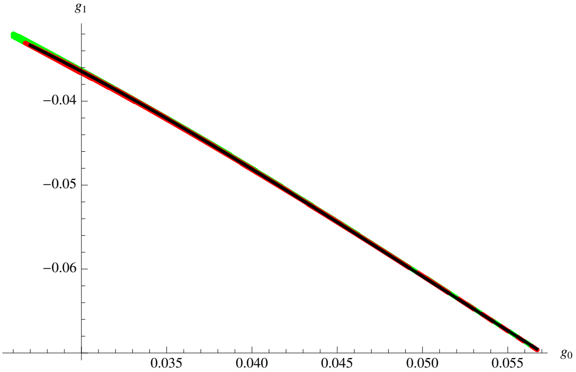
$$\varphi_i(r) = \sum_n g_n^{(i)} (r - r_{i,\mathrm{sing}})^n$$

- evaluate BCs
- initial conditions / matching



boundary condition at r_2

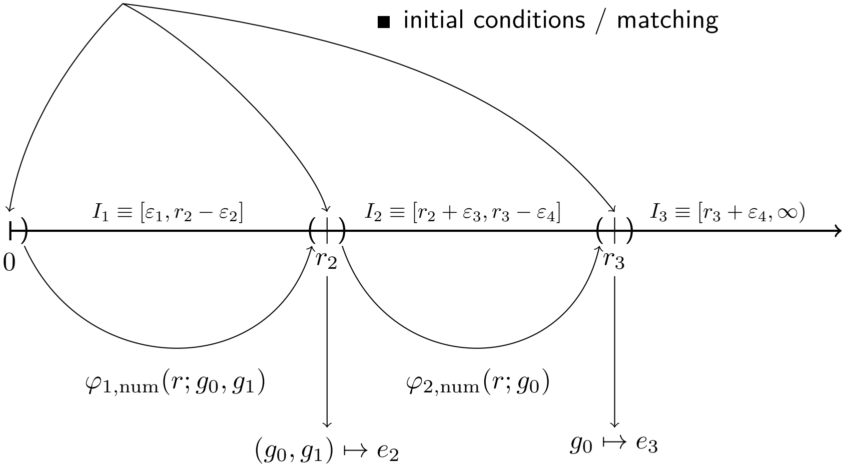
$e_2(g_0, g_1) < 0 \quad e_2(g_0, g_1) > 0 \quad e_2(g_0, g_1) \approx 0$



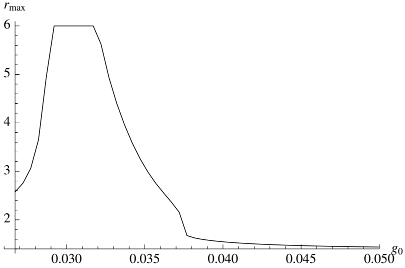
regular line: $g_0 \mapsto g_1(g_0)$

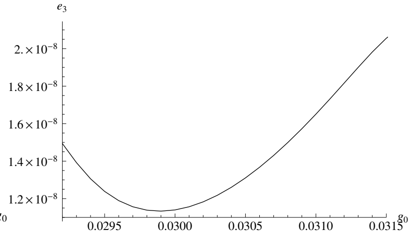
$$\varphi_i(r) = \sum_n g_n^{(i)} (r - r_{i,\text{sing}})^n$$

- evaluate BCs
- initial conditions / matching



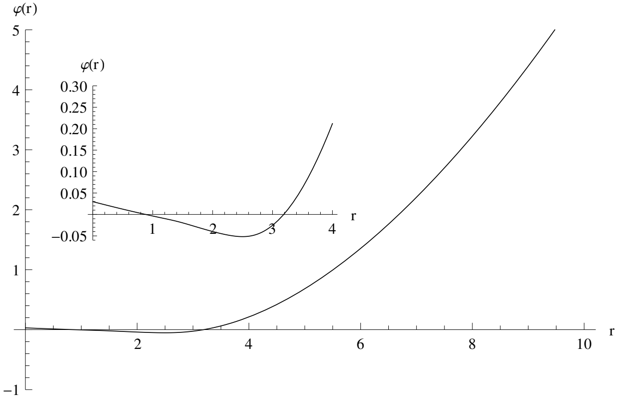
regularity at r_3





- movable singularities (correct r asymptotics of traces)
- e_3 possesses a minimum $\Longrightarrow$ one isolated fixed function

isolated fixed function



- extension of previous results: $\lambda_* = 0.411$, $g_* = 0.547$ and $\tau_* = 0.224$
- minimum at $r_{\mathrm{min}} \approx 2.500$

redundancy test [Dietz,Morris]

redundancy test function:

$$E_4(r) = 2\varphi(r) - r\varphi'(r)$$

proposed fixed function:

$$E_4(0) = 2g_0 > 0 \qquad E_4(r_{\min}) = 2\varphi(r_{\min}) < 0$$

E_4 is continuous $\implies$ E_4 has a zero for $r \in [0, r_{\min}]$

$\implies$ deformations of $\varphi(r)$ are *not redundant*, i.e. cannot be absorbed by a field redefinition

asymptotic behaviour for $r \to \infty$

rescaling: $r \equiv \varepsilon^{-a}\tilde{r}$ and $\varphi(r) \equiv \varepsilon^{-b}\tilde{\varphi}(\tilde{r})$ with $a, \varepsilon > 0$

At leading order in ε:

$$\frac{1}{\epsilon^b}\,\left(4\tilde{\varphi} - 2\tilde{r}\tilde{\varphi}'\right) = \frac{1}{\epsilon^{2a}}\,\frac{5\tilde{r}^3}{864\pi^2}\,\frac{2\,\tilde{r}\,\tilde{\varphi}'' - \tilde{\varphi}'}{\tilde{r}\,\tilde{\varphi}' + 6\,\tilde{\varphi}}$$

- classical scaling $(b > 2a)$: $\tilde{\varphi}(\tilde{r}) = A\tilde{r}^2 +$ subleading
- quantum scaling $(b < 2a)$: $\tilde{\varphi}(\tilde{r}) = A\tilde{r}^{3/2} + B$
- balanced scaling $(b = 2a)$: $\tilde{\varphi}(\tilde{r}) = A\tilde{r}^2\,(1 - \eta\log(\tilde{r}))$

numerical solution favors balanced scaling including non-analytic terms

construction principles for globally well-defined unique fixed function

- no topology change in the background (here four-sphere)
- no unphysical singularities (PI measure)
- exact heat-kernel (large r behavior)
- TT and scalar modes should be integrated out simultaneously (i.e. ELE)
- all quantum fluctuations need to be integrated out

summary and outlook

- geometric type flow equation
- unique and globally well-defined fixed function
- what are necessary conditions for globally well-defined fixed functions?
- more physical definition of fluctuations $g_{\mu\nu} = \bar{g}_{\mu\rho} \left(\mathrm{e}^{h}\right)^{\rho}{}_{\nu}$? [1507.00968]
- relation between singular structure and path integral measure?
- phenomenological implications of the fixed function?