Connections and geodesics in the space of metrics

The exponential parametrization from a geometric perspective

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The exponential metric parametrization

$$g_{\mu\nu} = \bar{g}_{\mu\rho}\big(\mathrm{e}^{h}\big)^{\rho}{}_{\nu}$$

($\bar{g}_{\mu\nu}$: background metric, $h_{\mu\nu} = h_{\nu\mu}$: symmetric tensor field)

Use in literature

- Kawai et al. ($\geq$ 1993): Perturbative QG in $d = 2 + \epsilon$
- Eichhorn (2013 and 2015): Unimodular QG
- A.N. (2014): Single & bi-metric EH truncations
- Codello and D'Odorico (2014): Scaling exponents and KPZ
- Percacci and Vacca (2015): Scalar tensor models
- Falls (2$\times$2015): Gauge independent EA at one-loop
- Labus, Percacci and Vacca (2015): Scalar tensor models
- Ohta and Percacci (2015): Conformal gravity
- Ohta, Percacci and Vacca (2015): $f(R)$ gravity
- Gies, Knorr and Lippoldt (2015): Analysis of parametrization dependence

The exponential metric parametrization

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Why is it used?

Up to now: viewed as appropriate choice of "coordinate system"

- Easy separation of conformal factor $\rightarrow$ trace of fluctuations $h_{\mu\nu} = \hat{h}_{\mu\nu} + \frac{1}{d}\bar{g}_{\mu\nu}\phi$, easy volume element $\sqrt{g} = \sqrt{\bar{g}}\,\mathrm{e}^{\frac{1}{2}\phi}$
- Avoid unphysical singularities in flow equations
- Reproduce central charge $c = 25$

Why do we care about parametrizations?

→ Because $\bar{g}_{\mu\rho}(\mathrm{e}^h)^\rho{}_\nu$ is a metric $\forall h_{\mu\nu}$ while $\bar{g}_{\mu\nu} + h_{\mu\nu}$ is not!

- $\bar{g}_{\mu\rho}(\mathrm{e}^h)^\rho{}_\nu$ is symmetric and has the same signature as $\bar{g}_{\mu\nu}$
- $\bar{g}_{\mu\nu} + h_{\mu\nu}$ is symmetric, but can have wrong signature, in particular it can be degenerate (e.g. for $h_{\mu\nu} = -\bar{g}_{\mu\nu}$)

Question: path integral $\int \mathcal{D}h$ over valid metrics only?

- Exponential parametrization respects nonlinear structure of space of metrics, $\int \mathcal{D}h$ involves only valid metrics
- Linear split (without further restrictions on $h_{\mu\nu}$): $\int \mathcal{D}h$ captures degenerate "metrics", too

Exp ↔ linear not a reparametrization! No on-shell equivalence!?

Overview

- View $h_{\mu\nu}$ as tangent vector and exponential parametrization as geodesics in the space of metrics $\rightarrow$ derive connection
- Show fundamental geometric origin of the connection
- Attention with Lorentzian signatures!
- Compare with Levi-Civita and Vilkovisky-DeWitt connection
- Covariance and geometric effective action

Derive connection in the space $\mathcal{F}$ of metrics

Connection s.t. geodesics in $\mathcal{F}$ are parametrized by $\bar{g}_{\mu\rho}(\mathrm{e}^{t\,h})^{\rho}{}_{\nu}$

Geodesic $t \mapsto g_{\mu\nu}(t)$ with BC $g_{\mu\nu}(0) = \bar{g}_{\mu\nu}$ and $g_{\mu\nu}(1) = g_{\mu\nu}$

$$\ddot{g}_{\mu\nu}(t) + \Gamma_{\mu\nu}^{\alpha\beta\,\rho\sigma}\,\dot{g}_{\alpha\beta}(t)\dot{g}_{\rho\sigma}(t) = 0 \qquad (*)$$

In Taylor series $g_{\mu\nu}(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \left(\frac{\mathrm{d}^n}{\mathrm{d}t^n} g_{\mu\nu}(t)\Big|_{t=0} \right)$ replace all higher derivatives in terms of $\dot{g}_{\mu\nu}(t)$ by eq. $(*)$ with $h_{\mu\nu} = \dot{g}_{\mu\nu}(0)$:

$$g_{\mu\nu} = g_{\mu\nu}(t=1) = \bar{g}_{\mu\nu} + h_{\mu\nu} - \frac{1}{2}\,\bar{\Gamma}_{\mu\nu}^{\alpha\beta\,\rho\sigma}\,h_{\alpha\beta}h_{\rho\sigma} + \mathcal{O}(h^3)$$

where $\bar{\Gamma}_{\mu\nu}^{\alpha\beta\,\rho\sigma} \equiv \Gamma_{\mu\nu}^{\alpha\beta\,\rho\sigma}(\bar{g})$

Derive connection in the space $\mathcal{F}$ of metrics

Compare this with direct expansion of $g_{\mu\nu} = \bar{g}_{\mu\rho}(\mathrm{e}^h)^\rho{}_\nu$

$$
\begin{aligned}
g_{\mu\nu} &= \bar{g}_{\mu\nu} + h_{\mu\nu} + \tfrac{1}{2}\, h_{\mu\rho}h^\rho{}_\nu + \mathcal{O}(h^3) \\
&= \bar{g}_{\mu\nu} + h_{\mu\nu} + \tfrac{1}{2}\, \delta^{(\alpha}_{(\mu}\, \bar{g}^{\beta)(\rho}\, \delta^{\sigma)}_{\nu)}\, h_{\alpha\beta}h_{\rho\sigma} + \mathcal{O}(h^3)
\end{aligned}
$$

where indices embraced by round brackets are symmetrized.

From 2nd order we read off (spacetime dependence restored):

$$
\Gamma^{\alpha\beta\,\rho\sigma}_{\mu\nu}(x,y,z) = -\delta^{(\alpha}_{(\mu}\, g^{\beta)(\rho}\, \delta^{\sigma)}_{\nu)}\, \delta(x-y)\delta(x-z)
$$

Derive connection in the space $\mathcal{F}$ of metrics

Yet to be proven: equality of expansions at all orders

(1) Insert new connection $\Gamma^{\alpha\beta\,\rho\sigma}_{\mu\nu} = -\delta^{(\alpha}_{(\mu}\, g^{\beta)(\rho}\, \delta^{\sigma)}_{\nu)}$ in geodesic equation:

$$\ddot{g}_{\mu\nu} - g^{\alpha\rho}\dot{g}_{\mu\alpha}\dot{g}_{\rho\nu} = 0$$

(2) Multiply with $g^{\nu\sigma}$, rewrite derivatives: $\frac{\mathrm{d}}{\mathrm{d}t}(\dot{g}_{\mu\nu}g^{\nu\sigma}) = 0$
$\Rightarrow \dot{g}_{\mu\nu}g^{\nu\sigma} = \mathrm{const}$

(3) This is a 1st order ODE: $\dot{g}_{\mu\nu}(t) = c^{\sigma}_{\mu}\, g_{\sigma\nu}(t)$

(4) Initial conditions: $g_{\mu\nu}(0) = \bar{g}_{\mu\nu}$ and $\dot{g}_{\mu\nu}(0) = c^{\sigma}_{\mu}\,\bar{g}_{\sigma\nu} = h_{\mu\nu}$

(5) Unique solution: matrix exponential

$$g_{\mu\nu}(t) = \bar{g}_{\mu\rho}(\mathrm{e}^{t\,h})^{\rho}{}_{\nu}$$

(6) Setting $t = 1$ proves the equality $\square$

The fundamental geometric origin of the connection

Pointwise character of geodesics (and $\Gamma^{\alpha\beta\,\rho\sigma}_{\mu\nu} \propto \delta(x-y)\delta(x-z)$)
$\Rightarrow$ Discussion reduces to 1 (arbitrary) spacetime point!

Locally metrics are symmetric matrices of prescribed signature:

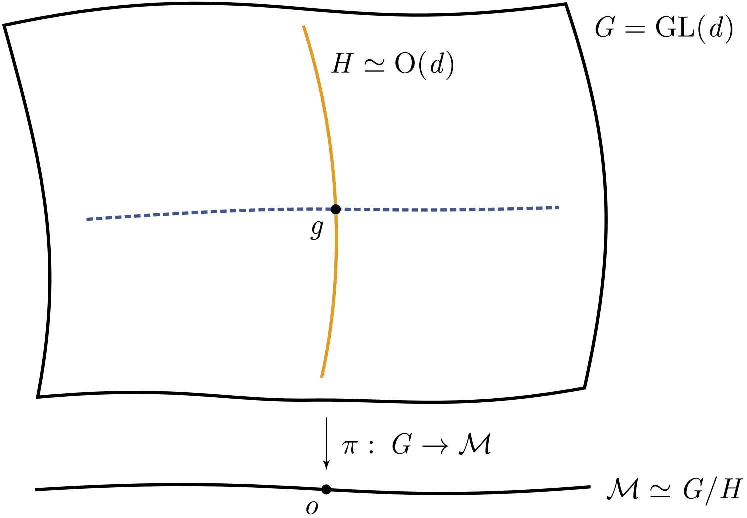
$$\mathcal{M} \equiv \left\{ A \in \mathrm{GL}(d) \middle|\, A^T = A,\ A \text{ has signature } (p,q) \right\}$$

For now: Euclidean signature (symmetric positive definite matrices)

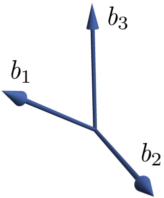
Agenda

- Show that $\mathcal{M}$ is base space of some principal bundle
- Principal bundle induces canonical connection

Illustration of the bundle



Relation between $G = \mathrm{GL}(d)$ and $\mathcal{M}$



- Fix metric, say η, by declaring some frame $B = (b_1 \ b_2 \ \dots \ b_d)$ to be orthonormal:

$$\eta(b_i, b_j) \equiv \eta_{\mu\nu}(b_i)^\mu (b_j)^\nu = \delta_{ij}$$

- In matrix form: $B^T \eta B = \mathbb{1}$, $B \in \mathrm{GL}(d)$

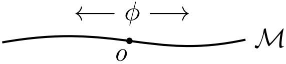
$$\eta = (B^{-1})^T \, \mathbb{1} \, B^{-1}$$

- But: invariance under $B \to BR^{-1}$ with $R \in \mathrm{O}(d)$
  $\Rightarrow$ coset space structure

$$\mathcal{M} \simeq \mathrm{GL}(d) / \, \mathrm{O}(d)$$

Group action and isotropy groups

Define group action of G on $\mathcal{M}$



$$\phi : G \times \mathcal{M} \to \mathcal{M}, \;\; (g, o) \mapsto \phi(g, o) \equiv (g^{-1})^T o \, g^{-1}$$

Consider fixed but arbitrary base point $\bar{o} \in \mathcal{M}$ (“origin”) with isotropy group (stabilizer)

$$H \equiv \left\{ h \in \mathbb{R}^{d \times d} \mid h^T \bar{o} h = \bar{o} \right\}$$

H is stabilizer since $\phi(h, \bar{o}) = \bar{o} \;\; \forall h \in H$

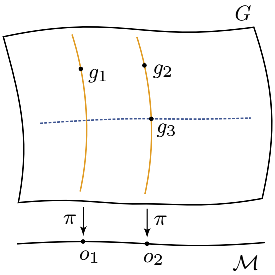
$\Rightarrow$ $\mathcal{M}$ is homogeneous space (i.e. coset space G/H without origin)

G as a principal bundle

Define canonical projection

$$\pi : G \to \mathcal{M}, \;\; g \mapsto \pi(g) \equiv (g^{-1})^T \bar{o} g^{-1}$$

$\Rightarrow (G, \pi, \mathcal{M}, H)$ is principal bundle



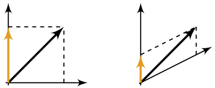
Tangent spaces: given by Lie algebras

$$\mathfrak{g} = \mathbb{R}^{d \times d}$$

$$\mathfrak{h} = \left\{ A \in \mathbb{R}^{d \times d} \,\middle|\, A^T \bar{o} = -\bar{o} A \right\} \quad \text{(vertical direction)}$$

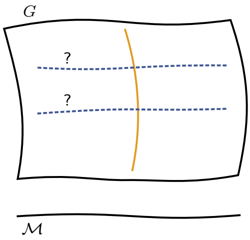
The canonical connection

What is horizontal in tangent space?
What about projections?



Projections depend on both coordinate axes!

Distinguished definition of horizontal direction



$$\mathfrak{m} = \left\{ A \in \mathbb{R}^{d \times d} \mid A^T \bar{o} = \bar{o} A \right\}$$

- $\mathfrak{m}$ is vector space complement of $\mathfrak{h}$ in $\mathfrak{g}$:
$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$$
- Both $\mathfrak{m}$ and $\mathfrak{h}$ are invariant under $\mathrm{Ad}(H)$
  $\Rightarrow$ Homogeneous space $\mathcal{M}$ is reductive
- By $\mathrm{d}\pi|_{\mathfrak{m}}$ we can identify $\mathfrak{m} \simeq T_{\bar{o}}\mathcal{M}$

Canonical connection determined by $\mathcal{H}_g \equiv \mathrm{d}L_g \mathfrak{m}$

Computation of the canonical connection

Metric on $\mathcal{M}$:

$$\gamma(X,\,Y) \equiv \mathrm{tr}(\bar{o}^{-1} X\, \bar{o}^{-1}\, Y) + \tfrac{c}{2}\, \mathrm{tr}(\bar{o}^{-1} X)\, \mathrm{tr}(\bar{o}^{-1}\, Y)$$

with $X, Y \in T_{\bar{o}}\mathcal{M}$ (symmetric matrices), c an arbitrary constant

- γ is G-invariant, i.e. the group action is isometric

Canonical connection on $(G, \pi, \mathcal{M}, H)$ induces connection on tangent bundle $T\mathcal{M} \simeq G \times_{\mathrm{Ad}(H)} \mathfrak{m} \equiv (G \times \mathfrak{m})/H$

- given by the Levi-Civita connection on $T\mathcal{M}$ w.r.t. γ:

$$\bar{\Gamma}(X,\,Y) = -\tfrac{1}{2}\big(X\,\bar{o}^{-1}\,Y + Y\,\bar{o}^{-1} X\big)$$

Index notation, $\bar{\Gamma}_{\mu\nu}^{\alpha\beta\,\rho\sigma}\, X_{\alpha\beta}\, Y_{\rho\sigma} \equiv \bar{\Gamma}(X,\,Y)$, base point $\bar{o} = \bar{g}_{\mu\nu}$:

$$\bar{\Gamma}_{\mu\nu}^{\alpha\beta\,\rho\sigma} = -\delta_{(\mu}^{(\alpha}\, \bar{g}^{\beta)(\rho}\, \delta_{\nu)}^{\sigma)}$$

Geodesics in $\mathcal{M}$

$\mathcal{M}$ inherits exponential map from $G = \mathrm{GL}(d)$ (matrix exponential)

$$\exp_{\bar{o}} X = \pi\left(\mathrm{e}^{\mathrm{d}\pi_e^{-1} X}\right)$$

for $X \in T_{\bar{o}}\mathcal{M}$. Inserting the canonical projection π:

$$o = \exp_{\bar{o}} X = \bar{o}\,\mathrm{e}^{\bar{o}^{-1} X}$$

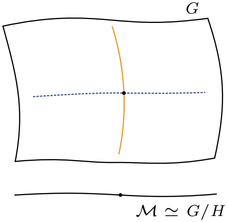
In index notation with $o = g_{\mu\nu}$, $\bar{o} = \bar{g}_{\mu\nu}$ and $X = h_{\mu\nu}$:

$$g_{\mu\nu} = \bar{g}_{\mu\rho} (\mathrm{e}^{h})^{\rho}{}_{\nu}$$

$\Rightarrow$ Geodesics in $\mathcal{M}$, right signature by construction!

Interim conclusion

- $\mathcal{M}$ is homogeneous space, $\mathcal{M} \simeq G/H$
- G has principal bundle structure
- Natural way of defining the horizontal direction $\Rightarrow$ canonical connection
- Geodesics in $\mathcal{M}$ w.r.t. canonical connection parametrized by $g_{\mu\nu} = \bar{g}_{\mu\rho}(\mathrm{e}^h)^\rho{}_\nu$



Exponential parametrization: adapted to basic structure of space of metrics

Two important terms for classifying $\mathcal{M}$

Geodesic completeness:

Every maximal geodesic is defined on the entire real line $\mathbb{R}$

- Geodesics "stay in $\mathcal{M}$" and do not run into singularities
- Exponential map defined on entire tangent space

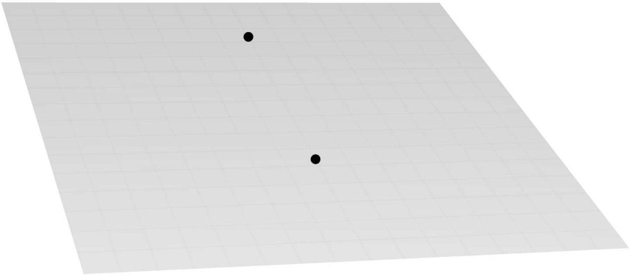
Geodesic connectedness:

Any two points in $\mathcal{M}$ can be connected by a geodesic

Note: Connectedness plus geodesic completeness does not imply geodesic connectedness!

Example 1: flat plane $\mathbb{R}^2$

Geodesically complete and geodesically connected



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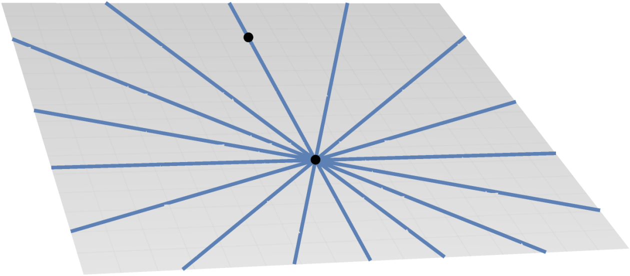
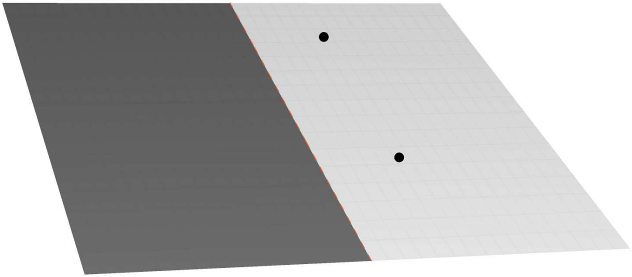


Example 2: half plane

Not geodesically complete but geodesically connected



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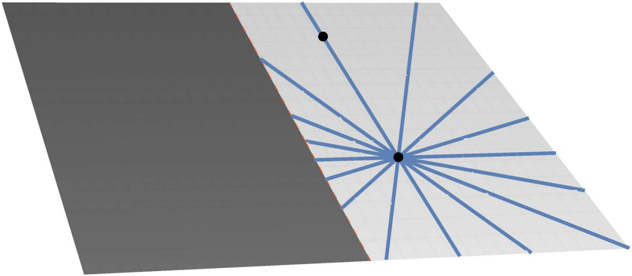
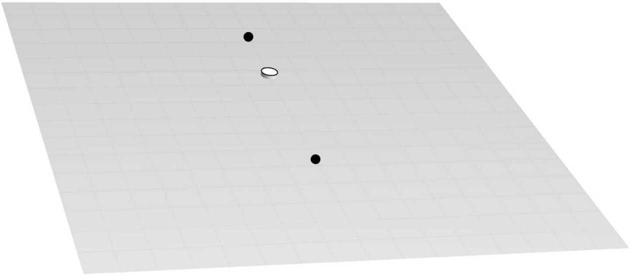


Example 3: punctured plane $\mathbb{R}\backslash\{0\}$

Neither geodesically complete nor geodesically connected



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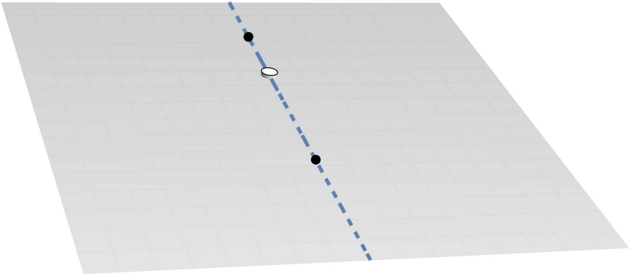
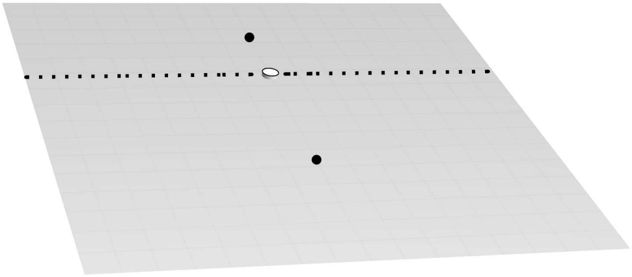


Example 4: punctured plane, non-flat connection

Geodesically complete but not geodesically connected



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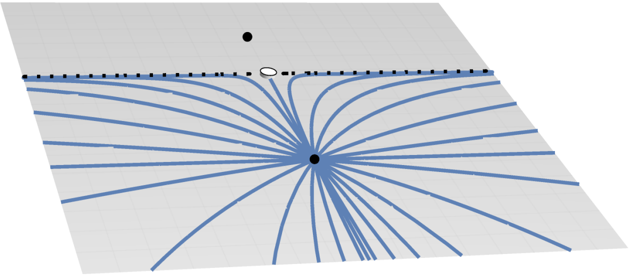

Properties of $\mathcal{M}$ for different signatures (p, q)

For all signatures (p, q) the set $\mathcal{M}$ is

- open ($\Rightarrow$ one chart sufficient)
- non-compact
- path-connected
- geodesically complete

For Euclidean signatures (p arbitrary, $q = 0$) $\mathcal{M}$ is

- geodesically connected
- simply connected

For Lorentzian signatures ($p \geq 1,\ q \geq 1$) $\mathcal{M}$ is

- not geodesically connected
- not simply connected

Illustration for 2×2-matrices

Parametrize symmetric matrices ($\supseteq \mathcal{M}$) by

$$\begin{pmatrix} z-x & y \\ y & z+x \end{pmatrix}$$

Eigenvalues given by

$$\lambda_{1,2} = z \pm \sqrt{x^2+y^2}$$

- Euclidean: $\lambda_1, \lambda_2 > 0$
  $\Rightarrow z > \sqrt{x^2+y^2}$
- Lorentzian: $\lambda_1 > 0$, $\lambda_2 < 0$
  $\Rightarrow -\sqrt{x^2+y^2} < z < \sqrt{x^2+y^2}$

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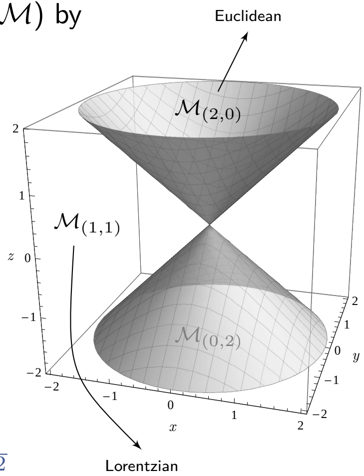
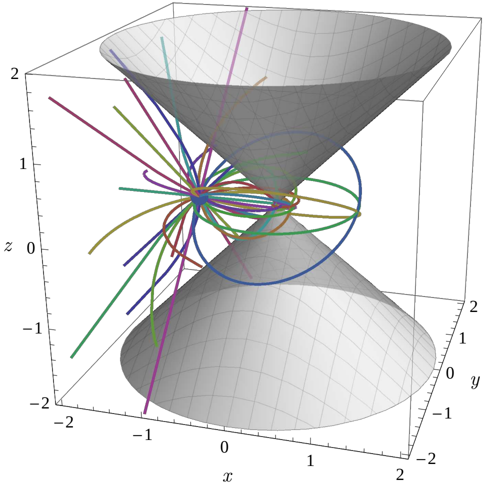


Geodesics in $\mathcal{M}_{(1,1)}$ (Lorentzian signature)



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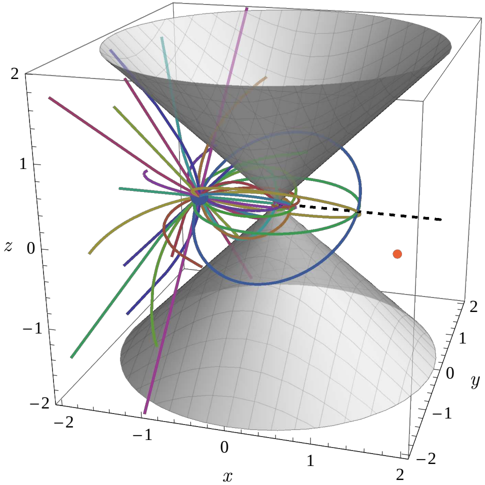
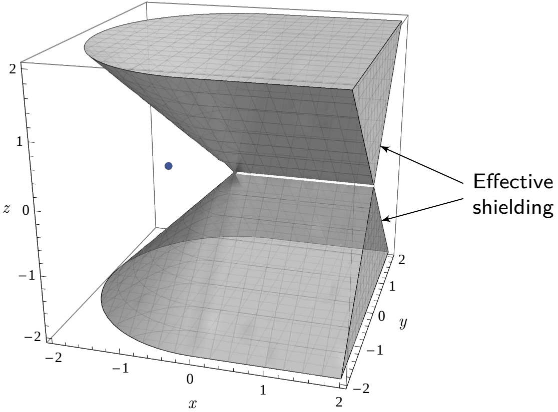


Region in $\mathcal{M}_{(1,1)}$ that can be reached by geodesics



Lorentzian case: exponential map neither surjective nor injective!

From $\mathcal{M}$ to $\mathcal{F}$

Recall: we had metric γ in $\mathcal{M}$. In index notation:

$$\gamma^{\mu\nu\,\rho\sigma} = g^{\mu(\rho}g^{\sigma)\nu} + \tfrac{c}{2}\, g^{\mu\nu} g^{\rho\sigma}$$

$\Rightarrow$ metric G in $\mathcal{F}$?

Yes. With correct spacetime dependence and density weight:

$$G^{\mu\nu\,\rho\sigma}(x,y) = \sqrt{\boldsymbol{g(x)}}\,\gamma^{\mu\nu\,\rho\sigma}(g(x))\delta(x-y)$$

This is the DeWitt metric. It is the unique metric that is

- ultra-local and diagonal in x-space
- gauge invariant (diffeomorphisms are isometric)

Connections on $\mathcal{M}$ and $\mathcal{F}$

Proportionality factor $\sqrt{g}$ entails further field dependence

$\Rightarrow$ Levi-Civita connection on $\mathcal{F}$ contains additional terms:

$$\Gamma_{\mathcal{F}}^{(\mathsf{LC})} = \left(\Gamma_{\mathcal{M}}^{(\mathsf{LC})} + T\right)(x)\,\delta(x-y)\delta(x-z)$$

General connection on $\mathcal{F}$: every smooth bi-linear bundle homomorphism A defines a connection by

$$\Gamma_{\mathcal{F}} = \Gamma_{\mathcal{F}}^{(\mathsf{LC})} + A$$

Choosing $A = -T$ cancels contributions from $\sqrt{g}$ and reproduces

$$\begin{aligned}\Gamma_{\mathcal{F}} &= \Gamma_{\mathcal{M}}^{(\mathsf{LC})}(x)\,\delta(x-y)\delta(x-z) \\ &= -\delta_{(\mu}^{(\alpha}\,g^{\beta)(\rho}(x)\,\delta_{\nu)}^{\sigma)}\,\delta(x-y)\delta(x-z)\end{aligned}$$

Connections on $\mathcal{M}$ and $\mathcal{F}$

Another famous choice is $A = A^{(\mathsf{VDW})}$ (Vilkovisky-DeWitt)

- adapted to gauge bundle structure of $\mathcal{F}$
- $A^{(\mathsf{VDW})}$ involves generators of gauge group
- highly non-local!

Summary:

$$\Gamma_{\mathcal{F}} = \Gamma_{\mathcal{F}}^{(\mathsf{LC})} + A$$

$A =$

0	LC	derived from metric	geodesics still calculable
$A^{(\mathsf{VDW})}$	VDW	adapted to gauge bundle structure	complicated non-local geodesics
$-T$	new	adapted to geometric structure of $\mathcal{M}$	very simple geodesics!

Covariance in field space $\mathcal{F}$

- Employ condensed DeWitt notation: $i \equiv (\mu\nu, x)$
- Consider a functional Γ of g and $\bar{g}$

$$\Gamma[g, \bar{g}]$$

- Parametrize g in terms of h by a geodesic: $g \equiv g[h; \bar{g}]$
- Define

$$\tilde{\Gamma}[h; \bar{g}] \equiv \Gamma\left[g[h; \bar{g}], \bar{g}\right]$$

- From geodesic equation follows

$$\boxed{\frac{\delta^n}{\delta h^{i_1} \cdots \delta h^{i_n}} \tilde{\Gamma}[h; \bar{g}]\Big|_{h=0} = \mathcal{D}_{(i_1} \cdots \mathcal{D}_{i_n)} \Gamma[g, \bar{g}]\Big|_{g=\bar{g}}}$$

$\Rightarrow$ Simple derivatives w.r.t. h are covariant derivatives in $\mathcal{F}$!

Covariance in field space $\mathcal{F}$

Consequences

- With the exponential parametrization, $\Gamma_k^{(2)}$ (appearing e.g. in the flow equation) is automatically covariant in field space:

$$\left.\frac{\delta^2\Gamma_k\left[\bar{g}\,\mathrm{e}^{\bar{g}^{-1}h},\bar{g}\right]}{\delta h^i \delta h^j}\right|_{h=0} = \mathcal{D}_{(i}\mathcal{D}_{j)}\,\Gamma_k[g,\bar{g}]\Big|_{g=\bar{g}}$$

- Geometric formalism (for any connection) suited for computing covariant objects
- Allows for construction of reparametrization invariant geometric effective action
- Modified Nielsen identities: relate $\delta\Gamma_k/\delta\bar{g} \leftrightarrow \delta\Gamma_k/\delta g$

Summary & conclusion

- Fundamental geometric structure $\mathcal{M} \simeq \mathrm{GL}(d)/\,\mathrm{O}(p,q)$
- Principal bundle induces canonical connection
- Geodesics parametrized by $g_{\mu\nu} = \bar{g}_{\mu\rho}(\mathrm{e}^h)^\rho{}_\nu$
- Produces only valid metrics!
- Attention with Lorentzian signatures