Solutions to the Reconstruction Problem in Asymptotic Safety

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Based on: T. R. Morris and Z. H. Slade, arXiv:1507.08657

Contents

The Reconstruction Problem

The effective average action UV cutoff $\Lambda >$ IR cutoff k

$$Z_k^\Lambda[J] = \int \mathcal{D}\phi\, \mathrm{e}^{-\mathcal{S}^{\mathrm{tot},\Lambda}[\phi]+\frac{1}{2}\phi\cdot\mathrm{R_k}\cdot\phi+\mathrm{J}\cdot\phi} \equiv \mathrm{e}^{\mathrm{W_k^\Lambda[J]}} \xrightarrow[\text{transform}]{\text{Legendre}} \hat{\Gamma}_\mathrm{k}^\Lambda[\varphi]$$

The flow equation

$$\partial_k \hat{\Gamma}_k^\Lambda[\varphi] = \frac{1}{2}\mathrm{tr}_\Lambda\left[\left(R_k + \hat{\Gamma}_k^{\Lambda(2)}\right)^{-1}\partial_k R_k\right]$$

Can remove UV cutoff Λ if $\partial_k R_k(p)$ decays quickly for $p^2 \gg k^2$:

$$\partial_k \hat{\Gamma}_k[\varphi] = \frac{1}{2}\mathrm{tr}\left[\left(R_k + \hat{\Gamma}_k^{(2)}\right)^{-1}\partial_k R_k\right]$$

- Search for asymptotically safe theory using $\hat{\Gamma}_k[\varphi]$.
- No need to define a regularized path integral and bare action.

The Reconstruction Problem

There are reasons for wanting access to path integral representation:

- Study Hamiltonian formulations.
- Understand properties of a the QFT (e.g. constraints, local symmetries, ...)
- Compare with other approaches (e.g. loop quantum gravity, perturbation theory, ...)

The reconstruction problem:

How can we reconstruct a bare action that corresponds to the asymptotically safe theory found using $\hat{\Gamma}_k$?

One-loop solution:

$$\hat{\Gamma}^{\Lambda}_{k=\Lambda}[\varphi] - \hat{\mathcal{S}}^{\Lambda}[\varphi] = \frac{1}{2}\mathrm{tr}_{\Lambda} \ln\left\{\hat{\mathcal{S}}^{\Lambda(2)}[\varphi] + R_{\Lambda}\right\}$$

E. Manrique and M. Reuter, arXiv:0811.3888

Definitions

Effective average action

$$\hat{\Gamma}_k[\varphi] = \frac{1}{2}\varphi \cdot p^2 \cdot \varphi + \Gamma_k[\varphi] \tag{1}$$

Total effective action

$$\begin{aligned}\Gamma_k^{\text{tot}}[\varphi] = & \quad \hat{\Gamma}_k[\varphi] + \frac{1}{2}\varphi \cdot R_k \cdot \varphi & (2) \\ = & \quad \Gamma_k[\varphi] + \frac{1}{2}\varphi \cdot (\Delta_k)^{-1} \cdot \varphi & (3)\end{aligned}$$

Multiplicative IR cutoff

$$\Delta_k = \frac{C_k(p)}{p^2} \tag{4}$$

such that

$$C_k(p) = \frac{p^2}{p^2 + R_k(p)} \tag{5}$$

Definitions

UV cutoff $\Lambda >$ IR cutoff k

$$Z_k^\Lambda[J] = \int \mathcal{D}\phi \, \mathrm{e}^{-\mathcal{S}^{\mathrm{tot},\Lambda}[\phi] + \frac{1}{2}\phi \cdot \mathrm{R_k} \cdot \phi + \mathrm{J} \cdot \phi} \tag{6}$$

UV & IR regularisation appear in resulting effective action:

$$\Gamma_k^{\mathrm{tot},\Lambda}[\varphi] = \Gamma_k^\Lambda[\varphi] + \frac{1}{2}\varphi \cdot (\Delta_k^\Lambda)^{-1} \cdot \varphi \tag{7}$$

Multiplicative cutoff

$$\Delta_k^\Lambda = \frac{C_k^\Lambda(p)}{p^2} \tag{8}$$

where e.g.

$$C_k^\Lambda(p) = \frac{p^2 \theta(\Lambda - |p|)}{p^2 + R_k(p)} \tag{9}$$

Definitions

Bare interactions

$$\hat{\mathcal{S}}^{\Lambda}[\phi] = \frac{1}{2}\phi \cdot p^2 \cdot \phi + \mathcal{S}^{\Lambda}[\phi]$$

Wilsonian interactions

$$\hat{S}^{k}[\Phi] = \frac{1}{2}\Phi \cdot p^2 \cdot \Phi + S^{k}[\Phi]$$

Total bare action

$$\mathcal{S}^{\mathrm{tot},\Lambda}[\phi] = \mathcal{S}^{\Lambda} + \frac{1}{2}\phi \cdot (\tilde{\Delta}^{\Lambda})^{-1} \cdot \phi$$

Total Wilsonian effective action

$$S^{\mathrm{tot},k}[\Phi] = S^{k}[\Phi] + \frac{1}{2}\Phi \cdot (\Delta^{k})^{-1} \cdot \Phi$$

where

$$\tilde{\Delta}^{\Lambda} = \frac{\tilde{C}^{\Lambda}(p)}{p^2}$$

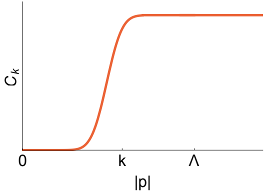
where

$$\Delta^{k} = \frac{C^{k}(p)}{p^2}$$

Definitions

Cutoff profiles:

(1)



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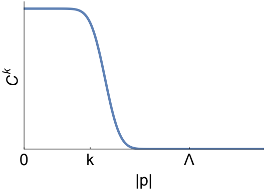

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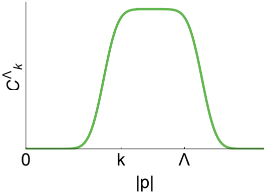

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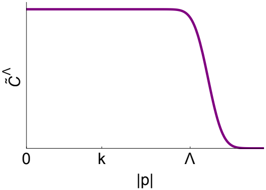


- $\tilde{C}^\Lambda$ and C^k obey the same general conditions but are otherwise unrelated.
- Remove UV cutoff: $C_k^\infty(p) \equiv C_k(p)$.

Definitions

Require that cutoffs obey **sum rule**

$$\boxed{C_k^\Lambda(p) + C^k(p) = \tilde{C}^\Lambda(p)} \tag{10}$$

Example:
$$\tilde{C}^\Lambda(p) = \theta(\Lambda - |p|), \quad C^k(p) = \theta(k - |p|) \tag{11}$$

$$\text{and} \quad C_k^\Lambda(p) = \theta(\Lambda - |p|) - \theta(k - |p|) \tag{12}$$

Compatibility condition:

$$\boxed{C_{k=\Lambda}^\Lambda(p) = 0} \tag{13}$$

If this is satisfied then

$$\tilde{C}^\Lambda(p) = C^{k=\Lambda}(p) \tag{14}$$

Solution 1

$\tilde{C}^\Lambda$

Starting point:

$$Z^\Lambda[J] = \int \mathcal{D}\phi \, \mathrm{e}^{-\mathcal{S}^{\mathrm{tot},\Lambda}[\phi]+J\cdot\phi} = \int \mathcal{D}\phi \, \mathrm{e}^{-\frac{1}{2}\phi\cdot\left(\tilde{\Delta}^\Lambda\right)^{-1}\cdot\phi-\mathcal{S}^\Lambda[\phi]+J\cdot\phi}$$

Introduce intermediate scale k by re-expressing propagator as

$$\tilde{\Delta}^\Lambda = \Delta_k^\Lambda + \Delta^k$$

Can show functional integral is

$$Z^\Lambda[J] = \int \mathcal{D}\phi_> \mathcal{D}\phi_< \, \mathrm{e}^{-\frac{1}{2}\phi_>\cdot\left(\Delta_k^\Lambda\right)^{-1}\cdot\phi_>-\frac{1}{2}\phi_<\cdot\left(\Delta^k\right)^{-1}\cdot\phi_<-\mathcal{S}^\Lambda[\phi_>+\phi_<]+J\cdot(\phi_>+\phi_<)}$$

- Drop constant of proportionality.
- Refer to $\phi_>$ ($\phi_<$) as high (low) modes.

Solution 1

Consider integral over high modes:

$$Z_k^\Lambda[J,\phi_<] \equiv \int \mathcal{D}\phi_> \, \mathrm{e}^{-\frac{1}{2}\phi_>\cdot\left(\Delta_k^\Lambda\right)^{-1}\cdot\phi_> - \mathcal{S}^\Lambda[\phi_>+\phi_<] + J\cdot(\phi_>+\phi_<)}$$

Perform linear shift $\phi_> = \phi - \phi_<$ and rewrite interaction $\mathcal{S}^\Lambda$ as a function of $\delta/\delta J$:

$$Z_k^\Lambda[J,\phi_<] = \mathrm{e}^{-\frac{1}{2}\phi_<\cdot\left(\Delta_k^\Lambda\right)^{-1}\cdot\phi_<} \, \mathrm{e}^{-\mathcal{S}^\Lambda[\frac{\delta}{\delta J}]} \int \mathcal{D}\phi \, \mathrm{e}^{-\frac{1}{2}\phi\cdot\left(\Delta_k^\Lambda\right)^{-1}\cdot\phi + \phi\cdot\left(J+\left(\Delta_k^\Lambda\right)^{-1}\cdot\phi_<\right)}$$

Perform another change of variables $\phi' = \phi - \Delta_k^\Lambda \cdot J - \phi_<$. Remaining integral is a decoupled Gaussian in ϕ'. After some rearranging:

$$Z_k^\Lambda[J,\phi_<] = \mathrm{e}^{\frac{1}{2}J\cdot\Delta_k^\Lambda\cdot J + J\cdot\phi_<} \, \mathrm{e}^{-\frac{1}{2}\left(J+\left(\Delta_k^\Lambda\right)^{-1}\cdot\phi_<\right)\cdot\Delta_k^\Lambda\cdot\left(J+\left(\Delta_k^\Lambda\right)^{-1}\cdot\phi_<\right)} \times$$
$$\mathrm{e}^{-\mathcal{S}^\Lambda[\frac{\delta}{\delta J}]} \, \mathrm{e}^{\frac{1}{2}\left(J+\left(\Delta_k^\Lambda\right)^{-1}\cdot\phi_<\right)\cdot\Delta_k^\Lambda\cdot\left(J+\left(\Delta_k^\Lambda\right)^{-1}\cdot\phi_<\right)}$$

Solution 1

Perform all derivatives in $\mathcal{S}^\Lambda[\delta/\delta J]$:

$$Z_k^\Lambda[J,\phi_<] = \mathrm{e}^{\frac{1}{2}J\cdot\Delta_k^\Lambda\cdot J + J\cdot\phi_< - S^k[\Delta_k^\Lambda\cdot J+\phi_<]} \tag{15}$$

Functional S^k we have introduced is the interaction part of a Wilsonian effective action. Why?

Substituting the above expression back into original partition function gives:

$$Z^\Lambda[J] = \int\mathcal{D}\phi_<\ \mathrm{e}^{-\frac{1}{2}\phi_<\cdot\left(\Delta^k\right)^{-1}\cdot\phi_< + \frac{1}{2}J\cdot\Delta_k^\Lambda\cdot J + J\cdot\phi_< - S^k[\Delta_k^\Lambda\cdot J+\phi_<]} \tag{16}$$

Consider that $\Delta_k^\Lambda\cdot J = 0$, then $Z^\Lambda[J]$ simplifies to

$$Z^\Lambda[J] = \int\mathcal{D}\phi_<\ \mathrm{e}^{-\frac{1}{2}\phi_<\cdot\left(\Delta^k\right)^{-1}\cdot\phi_< - S^k[\phi_<] + J\cdot\phi_<} \tag{17}$$

Solution 1

Integration over high modes related to generator of connected Green's functions W_k^Λ:

$$\mathrm{e}^{W_k^\Lambda[J,\phi_<]} \equiv Z_k^\Lambda[J,\phi_<] = \mathrm{e}^{\frac{1}{2}J\cdot\Delta_k^\Lambda\cdot J + J\cdot\phi_< - S^k[\Delta_k^\Lambda\cdot J+\phi_<]} \tag{18}$$

Legendre transform of W_k^Λ gives

$$\Gamma_k^{\mathrm{tot},\Lambda}[\varphi,\phi_<] = -W_k^\Lambda[J,\phi_<] + J\cdot\varphi \tag{19}$$

$$= \frac{1}{2}(\varphi-\phi_<)\cdot\left(\Delta_k^\Lambda\right)^{-1}\cdot(\varphi-\phi_<) + \Gamma_k^\Lambda[\varphi] \tag{20}$$

From (18), (19) and (20):

$$\boxed{S^k[\Phi] = \Gamma_k^\Lambda[\varphi] + \frac{1}{2}(\varphi-\Phi)\cdot(\Delta_k^\Lambda)^{-1}\cdot(\varphi-\Phi)} \tag{21}$$

where $\Phi = \Delta_k^\Lambda \cdot J + \phi_<$.

Solution 1

Duality relation:

$$\boxed{S^k[\Phi] = \Gamma_k^\Lambda[\varphi] + \frac{1}{2}(\varphi - \Phi) \cdot (\Delta_k^\Lambda)^{-1} \cdot (\varphi - \Phi)} \tag{22}$$

If we have $\{\Gamma_k, 0 \leq k < \infty\}$, then $\Gamma_k \equiv \Gamma_k^\infty$, $\Delta_k \equiv \Delta_k^\infty$ and

$$\boxed{S^k[\Phi] = \Gamma_k[\varphi] + \frac{1}{2}(\varphi - \Phi) \cdot (\Delta_k)^{-1} \cdot (\varphi - \Phi)} \tag{23}$$

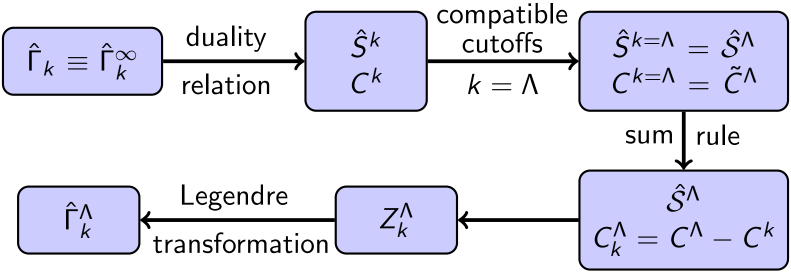
- S^k with $C^k = 1 - C_k$ can play the role of the bare interactions: $S^{k=\Lambda} = \mathcal{S}^\Lambda$.
- Can reconstruct partition function Z^Λ (fully regulated in UV).
- Not the same Z_k^Λ from whence Γ_k (Γ_k^Λ) came!

Solution 2

What hope is there of finding an exact relationship between $\hat{\Gamma}_k$ and the bare action which lives inside original functional integral?

$$\hat{\Gamma}_k \xrightarrow{?} Z_k^{\Lambda} = \int \mathcal{D}\phi \, \mathrm{e}^{-\mathcal{S}^{\mathrm{tot},\Lambda}[\phi] + \frac{1}{2}\phi \cdot \mathrm{R_k} \cdot \phi + \mathrm{J} \cdot \phi}$$

- $\hat{\Gamma}_k^{\Lambda}$ derived from Z_k^{Λ} necessarily depends on both cutoffs Λ and k.
- Best we can hope for is map from $\hat{\Gamma}_k$ to pair $\{\mathcal{S}^{\Lambda}, \hat{\Gamma}_k^{\Lambda}\}$ s.t. $\hat{\Gamma}_k^{\Lambda} \to \hat{\Gamma}_k$ as $\Lambda \to \infty$.



Solution 2

Can construct Γ_k^Λ vertex by vertex from Γ_k by utilising the duality relation:

$$S^k[\Phi] = \Gamma_k^\Lambda[\varphi] + \tfrac{1}{2}(\varphi - \Phi)\cdot(\Delta_k^\Lambda)^{-1}\cdot(\varphi - \Phi) \tag{24}$$

$$= \Gamma_k^{\mathring{\Lambda}}[\mathring{\varphi}] + \tfrac{1}{2}(\mathring{\varphi} - \Phi)\cdot(\Delta_k^{\mathring{\Lambda}})^{-1}\cdot(\mathring{\varphi} - \Phi) \tag{25}$$

where we have two different UV cutoff profiles $\tilde{C}^\Lambda$ and $\mathring{C}^{\mathring{\Lambda}}$ with $\mathring{\Lambda} > \Lambda$,

$$\implies \Gamma_k^\Lambda[\varphi] = \Gamma_k^{\mathring{\Lambda}}[\mathring{\varphi}] + \frac{1}{2}(\mathring{\varphi} - \Phi)\cdot(\Delta_\Lambda^{\mathring{\Lambda}})^{-1}\cdot(\mathring{\varphi} - \Phi) \tag{26}$$

Send $\mathring{\Lambda} \to \infty$

$$\implies \Gamma_k^\Lambda[\Phi] = \Gamma_k[\varphi] + \frac{1}{2}(\varphi - \Phi)\cdot(\Delta_\Lambda)^{-1}\cdot(\varphi - \Phi) \tag{27}$$

- Relation between two effective actions with different UV cutoffs.
- Change from $\tilde{C}^\Lambda$ to $\mathring{C}^{\mathring{\Lambda}}$ implies change $\mathcal{S}^\Lambda \mapsto \mathring{\mathcal{S}}^{\mathring{\Lambda}}$.

Solution 2

How does solution 2 relate to the one-loop solution?

$$\hat{\Gamma}^{\Lambda}_{k=\Lambda}[\varphi] - \hat{\mathcal{S}}^{\Lambda}[\varphi] = \frac{1}{2}\mathrm{tr}_{\Lambda} \ln\Big\{\hat{\mathcal{S}}^{\Lambda(2)}[\varphi] + R_{\Lambda}\Big\} \tag{28}$$

E. Manrique, M. Reuter, arXiv:0811.3888

Re-express (28) as

$$\Gamma^{\Lambda}_{k=\Lambda}[\varphi] = \mathcal{S}^{\Lambda}[\varphi] + \frac{1}{2}\mathrm{tr} \ln\Big\{\mathcal{S}^{\Lambda(2)}[\varphi] + (\Delta^{\Lambda}_{\Lambda})^{-1}\Big\} \tag{29}$$

In solution 2 we specialise to compatible cutoffs: $C^{\Lambda}_{k=\Lambda}(p) = 0$.

$$\implies \Gamma^{\Lambda}_{k=\Lambda}[\varphi] = \mathcal{S}^{\Lambda}[\varphi] \tag{30}$$

$$\mathrm{tr} \ln\Big\{\mathcal{S}^{\Lambda(2)}[\varphi] + (\Delta^{\Lambda}_{\Lambda})^{-1}\Big\} = -\mathrm{tr} \ln\Big\{\Delta^{\Lambda}_{\Lambda}\Big\} + \mathrm{tr} \ln\Big\{1 + \Delta^{\Lambda}_{\Lambda} \cdot \mathcal{S}^{\Lambda(2)}[\varphi]\Big\}$$

Solution 2

Most direct expression relating Wilsonian interactions and bare interactions:

$$Z_k^\Lambda[0, \phi_<] = e^{-S^k[\phi_<]} = \int \mathcal{D}\phi_> \mathrm{e}^{-\frac{1}{2}\phi_> \cdot (\Delta_\mathrm{k}^\Lambda)^{-1} \cdot \phi_> - \mathcal{S}^\Lambda[\phi_> + \phi_<]} \tag{31}$$

For compatible cutoffs at $k = \Lambda$

$$S^\Lambda[\varphi] = \mathcal{S}^\Lambda[\varphi] \tag{32}$$

Triple equality:

$$\Gamma^\Lambda_{k=\Lambda}[\varphi] = \mathcal{S}^\Lambda[\varphi] = S^\Lambda[\varphi] \tag{33}$$

- For non-compatiblle cutoffs there is still a non-trivial functional integral to do at $k = \Lambda$.

Summary and Conclusions

- Provided two solutions to the reconstruction problem:

(1) Exact duality relation

$$\boxed{S^k[\Phi] = \Gamma_k[\varphi] + \frac{1}{2}(\varphi - \Phi)\cdot(\Delta_k)^{-1}\cdot(\varphi - \Phi)}$$

(2) Map from $\hat{\Gamma}_k$ to pair $\{\hat{\mathcal{S}}^\Lambda, \hat{\Gamma}_k^\Lambda\}$ s.t. $\hat{\Gamma}_k^\Lambda \to \hat{\Gamma}_k$ as $\Lambda \to \infty$

$$\begin{tikzcd}
\hat{\Gamma}_k \equiv \hat{\Gamma}_k^\infty \arrow[r, "\text{duality}", "\text{relation}"'] & \begin{array}{c} \hat{S}^k \\ C^k \end{array} \arrow[r, "\begin{array}{c}\text{compatible}\\ \text{cutoffs}\end{array}", "k=\Lambda"'] & \begin{array}{c} \hat{S}^{k=\Lambda} = \hat{\mathcal{S}}^\Lambda \\ C^{k=\Lambda} = \tilde{C}^\Lambda \end{array} \arrow[d, "\text{sum}"', "\text{rule}"] \\
\hat{\Gamma}_k^\Lambda & Z_k^\Lambda \arrow[l, "\text{Legendre}"', "\text{transformation}"] & \begin{array}{c} \hat{\mathcal{S}}^\Lambda \\ C_k^\Lambda = C^\Lambda - C^k \end{array} \arrow[l]
\end{tikzcd}$$

Summary and Conclusions

- Relation between two effective average actions with different UV cutoff profiles $\tilde{C}^{\Lambda}$, $\mathring{C}^{\mathring{\Lambda}}$:

$$\Gamma_k^{\Lambda}[\varphi] = \Gamma_k^{\mathring{\Lambda}}[\mathring{\varphi}] + \frac{1}{2}(\mathring{\varphi} - \Phi) \cdot (\Delta_{\Lambda}^{\mathring{\Lambda}})^{-1} \cdot (\mathring{\varphi} - \Phi)$$

- Explored relation to one-loop formula.
- Generalise relations to multiple fields, fermions, ...
- Extend to full quantum gravity.

Additional slide: Why a UV regulated effective average action must depend on the UV regulator

Re-write flow equation

$$\frac{\partial}{\partial k}\hat{\Gamma}_k^{\Lambda}[\varphi] = \frac{1}{2}\mathrm{tr}_{\Lambda}\left[\left(R_k + \frac{\delta^2\hat{\Gamma}_k^{\Lambda}}{\delta\varphi\delta\varphi}\right)^{-1}\frac{\partial R_k}{\partial k}\right] \tag{34}$$

$$= \frac{1}{2}\mathrm{tr}\left[\left(R_k + \frac{\delta^2\hat{\Gamma}_k^{\Lambda}}{\delta\varphi\delta\varphi}\right)^{-1}\frac{\partial R_k}{\partial k}\right] - \frac{1}{2}\mathrm{tr}\left[\theta(|p|-\Lambda)\left(R_k + \frac{\delta^2\hat{\Gamma}_k^{\Lambda}}{\delta\varphi\delta\varphi}\right)^{-1}\frac{\partial R_k}{\partial k}\right] \tag{35}$$

(with Λ sharp)

- Remainder term vanishes for optimised IR cutoff profle $\partial_k R_k(p) = 2k\theta(k^2 - p^2)$.
- $\hat{\Gamma}_k^{\Lambda}$ obeys same FRGE as $\hat{\Gamma}_k$ for $k \leq \Lambda$? Consistently set $\hat{\Gamma}_k^{\Lambda} = \hat{\Gamma}_k$?

Additional slide: Why a UV regulated effective average action must depend on the UV regulator

Full inverse propagator:

$$\hat{\Delta}^{-1}(p) := R_k(p) + \left.\frac{\delta^2 \hat{\Gamma}_k^{\Lambda}}{\delta\varphi(p)\delta\varphi(-p)}\right|_{\varphi=0} \tag{36}$$

Define $\Gamma'[\varphi]$ to be the remainder after the term quadratic in the fields is removed. Then

$$\left(R_k + \frac{\delta^2 \hat{\Gamma}_k^{\Lambda}}{\delta\varphi\delta\varphi}\right)^{-1}(p,-p) = \left(\hat{\Delta}^{-1} + \frac{\delta^2 \Gamma'}{\delta\varphi\delta\varphi}\right)^{-1}(p,-p) \tag{37}$$

$$\begin{aligned} &= \hat{\Delta}(p) - \hat{\Delta}(p)\frac{\delta^2\Gamma'}{\delta\varphi(p)\delta\varphi(-p)}\hat{\Delta}(p) \\ &+ \int^{\Lambda}\frac{d^d q}{(2\pi)^d}\,\hat{\Delta}(p)\frac{\delta^2\Gamma'}{\delta\varphi(p)\delta\varphi(-p-q)}\hat{\Delta}(p+q)\frac{\delta^2\Gamma'}{\delta\varphi(p+q)\delta\varphi(-p)}\hat{\Delta}(p) - \cdots . \end{aligned}$$

Additional slide: Why a UV regulated effective average action must depend on the UV regulator

The momentum q is the external momentum injected by the fields remaining in Γ':

$$\frac{\delta^2 \Gamma'}{\delta\varphi(p)\delta\varphi(-p-q)} = \Gamma^{(3)}(p, -p-q, q; k, \Lambda)\varphi(-q) + \mathcal{O}(\varphi^2) \quad (38)$$

- External momenta $|q| \leq \Lambda$ and momenta running through internal line $|p+q| \leq \Lambda$ are restricted.
- Momentum p already cutoff at k due to $\partial_k R_k(p)$ and so Λ invisible for it.
- However cutoff Λ remains inside inverse because of internal momenta.
- $\hat{\Gamma}_k^\Lambda$ non-trivial function of Λ.