A renormalization group approach to multi-field models

arXiv:1510.04807

see also: arXiv:1407.7442, arXiv:1306.2952

David Mesterházy

Albert Einstein Center for Fundamental Physics, Universität Bern

In collaboration with:

Astrid Eichhorn, Thomas Helfer (Imperial College London),

and Michael M. Scherer (Universität Heidelberg)

Asymptotic Safety seminar

December 14, 2015

Introduction

Properties of interacting fixed points and RG trajectories for multi-field theories in $d = 3$ Euclidean dimensions.

We consider coupled vector representations of $O(N)$ symmetry group, i.e., $\oplus_i O(N_i)$ theories. This allows for a large number of possible scaling solutions!

These theories are trivial in $d = 4$ and therefore serve only as a toy model for asymptotically safe and / or IR-complete theories.

. . . but nevertheless might be useful for coupling scalar fields to gravity in $d = 4$ dimensions.

Renormalization group approach

To establish the scaling properties of fixed point solutions in $d = 3$ dimensions we use a Wilsonian RG framework (functional RG / ϵ-expansion).

In the absence of symmetry-breaking fields, all FPs arise from a perturbation of the Gaussian scaling solution.

The upper critical dimension of the considered models is $d = 4$ and therefore a low-order expansion in $\epsilon = 4 - d$ is not applicable (although it might be useful to inspect the presence of different FPs).

Truncations of functional RG flow equations for multi-field models

The RG β-functions are derived from the nonperturbative flow equation for the scale-dependent effective action functional $\Gamma = \Gamma\left[\{\phi_i\}\right]$ in d-dimensional Euclidean space:

$$k \frac{\partial}{\partial k} \Gamma = \mathrm{Flow}_R\left[\Gamma^{(2)}\right]$$

Wetterich, PLB **301** (1993) 90

k is a characteristic (RG) scale associated with the regulator function $R \equiv R(k)$.

The infinite hierarchy of flow equations is truncated by the following *ansatz* for the effective action of n distinct field degrees of freedom ϕ_i:

$$\Gamma = \int d^d x \left[\frac{1}{2} \sum_{i=1}^{n} Z_i \left(\partial \phi_i\right)^2 + U(\{\phi_i\}) \right]$$

$$U = \sum_{m_1+\ldots+m_n=2}^{M} \lambda_{m_1 \cdots m_n} \frac{\prod_{i=1}^{n} \left(\phi_i^2 - \kappa_i\right)^{m_i}}{\prod_{i=1}^{n} m_i!},$$

Although a large number of couplings $\lambda_{m_1 \cdots m_n}$ and parameters are introduced, typically only few of them appear in the bare action $S = \Gamma(k = \Lambda)$ defined at the cutoff scale.

Questions?

We introduce the following notation for the β-functions

$$\beta_{m_1\cdots m_n} \equiv k\frac{\partial\lambda_{m_1\cdots m_n}}{\partial k}$$

that are expressed in terms of the rescaled (renormalized, dimensionless) couplings:

$$\lambda_{m_1\cdots m_n} \to K_d \prod_{i=1}^{n} Z_i^{-2m_i} k^{(d-4)m_i} \lambda_{m_1\cdots m_n}$$

$$K_d = \left[(4\pi)^{d/2}\Gamma(d/2+1)/2\right]^{-1}$$

They take the following form:

$$\beta_{m_1\cdots m_n} = \underbrace{-d_{m_1\cdots m_n}\lambda_{m_1\cdots m_n}}_{\text{canonical scaling}} + k^{-d_{m_1\cdots m_n}} \underbrace{L(\{\lambda_{m'_1\cdots m'_n}\})}_{\text{loop contributions}}$$

$$\text{Critical exponents:} \quad \theta \in -\mathrm{Spec}\left(\frac{\partial\beta_M}{\partial\lambda_{M'}}\right)_{\mathrm{FP}}$$

In the following, we will assume that the parameters κ_i have been tuned to their respective critical values (for a given FP).

In general, multi-field models will feature a large number of possible scaling solutions.

The β-functions are non-polynomial functions of the couplings and it is therefore challenging to make sure that numerical fixed-point searches do indeed uncover all stable fixed points of the system.

How can we classify possible solutions and identify stability properties?

To address this problem, we match the solutions of the renormalization group β-functions derived within the framework of the functional RG to those obtained with the Wilsonian momentum-shell RG by employing an expansion in $\epsilon = 4 - d$.

The β-functions to one-loop order in the ϵ-expansion may also be obtained from the nonperturbative RG flow equation.

This is achieved by employing an expansion around the upper critical dimension and restricting the functional space to those operators that appear in the bare action.

By virtue of one-loop universality, the expanded functional RG β-functions are exact and independent of the chosen (nonperturbative) regulator.

We obtain:

$$\beta^{\text{1-loop}}_{m_1\cdots m_n} = \underbrace{-d_{m_1\cdots m_n}(\epsilon)\lambda_{m_1\cdots m_n}}_{\text{canonical scaling}} + k^{-d_{m_1\cdots m_n}} \underbrace{L^{\text{1-loop}}(\{\lambda_{m'_1\cdots m'_n}\})}_{\text{loop contributions}} + \mathcal{O}(\epsilon^3)$$

$$\lambda_{m_1\cdots m_n} = \mathcal{O}(\epsilon)$$

$$\epsilon = 4 - d$$

Questions?

In the context of coupled-field models one finds that different fixed points govern the IR scaling behavior of the model.

First we examine (multicritical) scaling solutions of the $O(N_1) \oplus O(N_2)$ theory.

In this case, two relevant (mass-like) parameters need to be tuned to put the system onto the critical hypersurface of a given FP. Note, that the number of parameters that need to be tuned increases with the number of coupled vector representations.

$O(N_1) \oplus O(N_2)$ two-field model

The decoupled (DFP), isotropic fixed point (IFP), and partially decoupled scaling solutions, can be deduced from the existence of the Wilson-Fisher fixed point.

They correspond to a complete decoupling of the two sectors with a discrete Z_2 exchange symmetry, a full symmetry enhancement to the $O(N_1 + N_2)$ symmetry group, and partial decoupling, respectively.

This model features another fully-coupled FP – the biconal fixed point (BFP) – that emerges due to the nontrivial interactions between the two field sectors (no symmetry enhancement).

In general, the presence of a large number of nontrivial FPs provides the possibility to define complete RG trajectories, with interacting FPs both in the IR / UV.

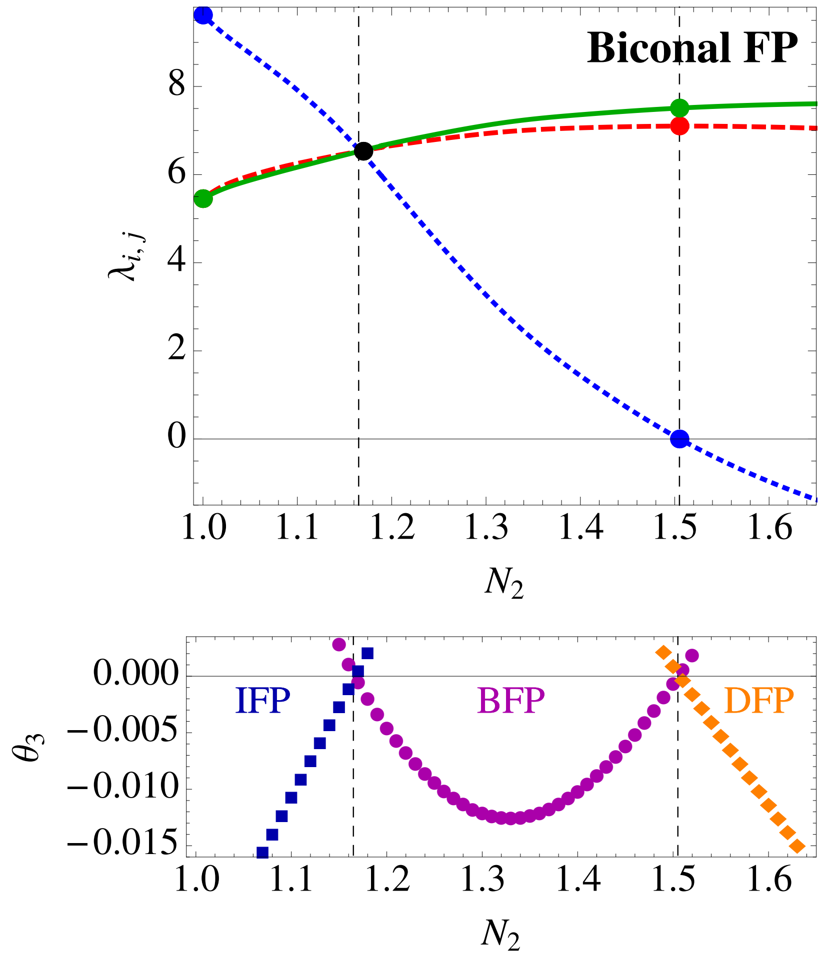
Stability exchange in two-field models

Keep $N_1 = 1$ fixed.

As we increase N_2 to larger values, the Biconal FP exchanges stability between two symmetry-enhanced FPs.

The stability properties of the respective scaling solutions is captured be the exponent θ_3 (θ_1 and θ_2 relate to the relevant κ_i parameters).

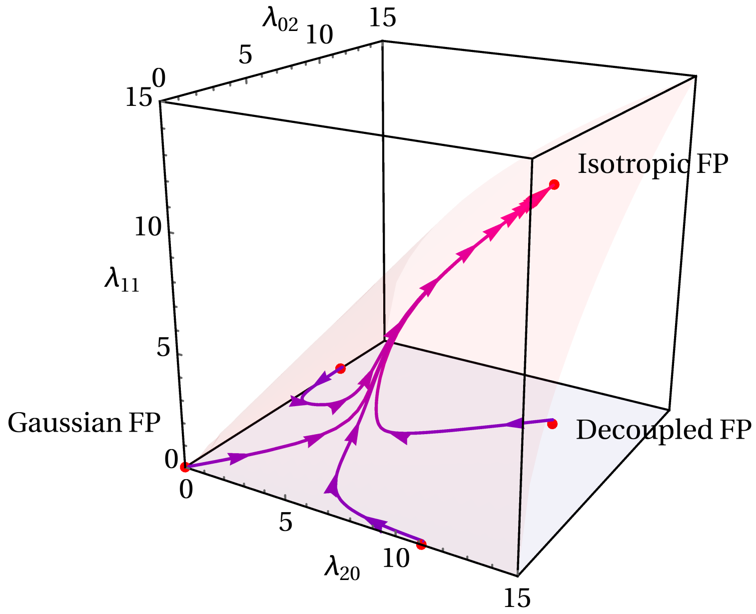
All possible multicritical FPs can be reached.



Example for complete RG trajectories with continuous symmetry enhancement.

On critical hypersurface RG flow is attracted to IR-stable fixed point with $O(2)$ symmetry.

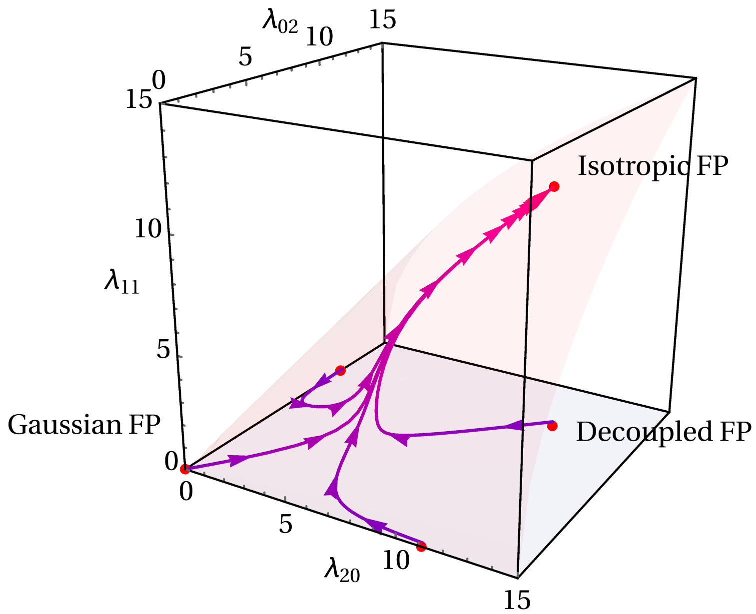
Two-field model, functional RG ($d = 3$)
$N_1 + N_2 = 2$

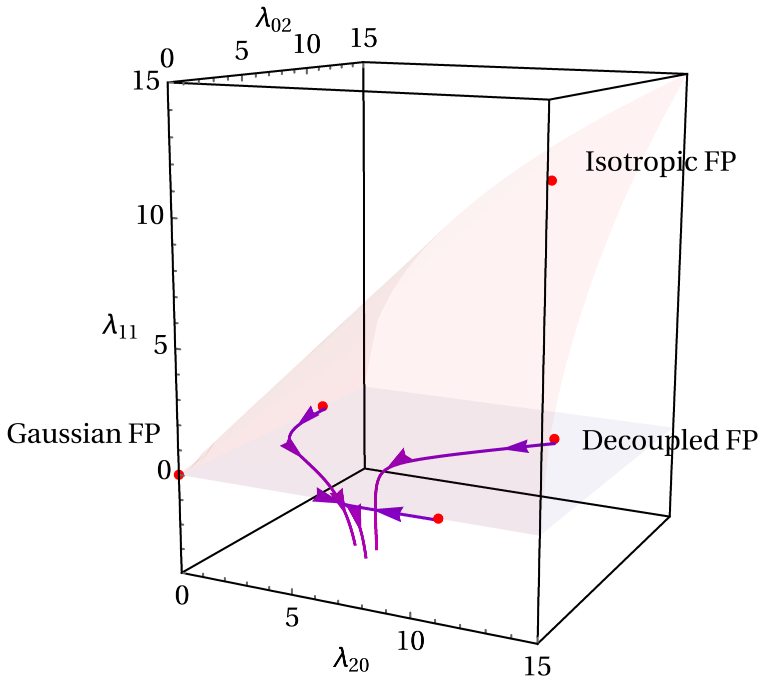


Example for complete RG trajectories with continuous symmetry enhancement.

On critical hypersurface RG flow is attracted to IR-stable fixed point with $O(2)$ symmetry – if initial couplings are within the domain of attraction.

Two-field model, functional RG ($d = 3$)
$N_1 + N_2 = 2$





Questions?

IR-stability of two-field models in (N_1, N_2) plane

Functional RG ($d = 3$) to different orders in the field expansion:

Isotropic FP is IR-stable only for small

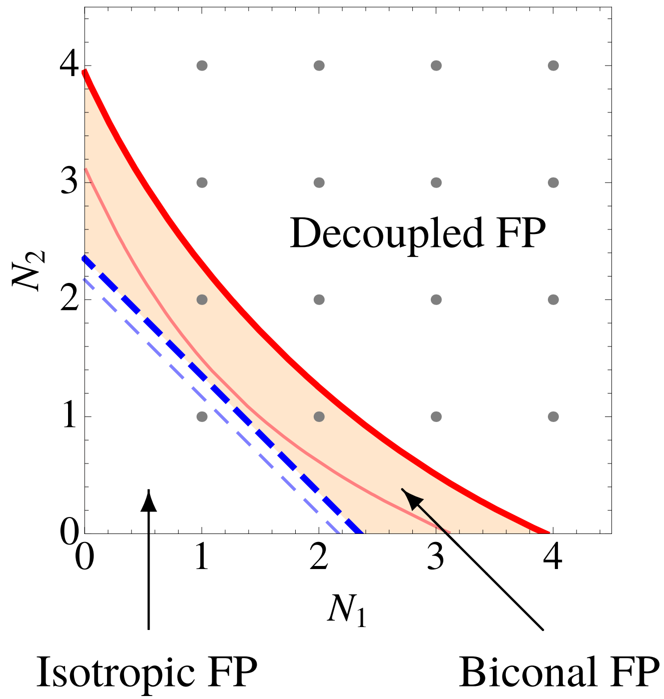
$$N_1 + N_2 \lesssim 2.4$$

Decoupled FP takes over IR-stability for

$$N_1 + N_2 \gtrsim 4$$

which can also be deduced from the scaling relation:

$$\theta_3 = \theta_1 + \theta_2 - d$$



These results are comparable to those obtained with the fourth order ϵ-expansion and six-loop fixed dimension expansion ($d = 3$).

Calabrese *et al.*, PRB **67** (2003) 054505

η-conjecture

"In general Φ^4 theories with a single quadratic invariant, the infrared stable FP is the one that corresponds to the fastest decay of correlations."

Vicari & Zinn-Justin, NJP **8** (2006) 321

This suggests that all one needs to do to assess IR-stability is to check the values of the anomalous dimensions η (if an IR-stable FP exists).

Can this conjecture be extended to multi-field theories?

"In this situation, the exponent η is replaced by a matrix and the conjecture applies to the trace of the matrix. However, the empirical evidence, beyond the ϵ-expansion is more limited."

While the present data is consistent with this statement, more elaborate truncations are necessary to make a definite statement.

Questions?

So how do these results generalize to multi-field models?

We systematically investigate the properties of scaling solutions and their respective domains of attraction. This allows us to define complete RG trajectories for these models.

Due to the complicated non-polynomial nature of the RG β-functions we start by considering a one-loop ϵ-expansion to identify all possible FPs that arise from the perturbation around the Gaussian FP.

Here, three relevant (mass-like) parameters need to be tuned to put the system onto the critical hypersurface.

One-loop ϵ-expansion and three-field models

Although the β-functions take a simple form, they still allow for a large number of FPs . . .
To one-loop order in the ϵ-expansion, the β functions take the form:

$$\begin{aligned}
\beta_{200} &= -\epsilon\lambda_{200} + (N_1+8)\lambda_{200}^2 + N_2\lambda_{110}^2 + N_3\lambda_{101}^2 \\
\beta_{020} &= -\epsilon\lambda_{020} + N_1\lambda_{110}^2 + (N_2+8)\lambda_{020}^2 + N_3\lambda_{011}^2 \\
\beta_{002} &= -\epsilon\lambda_{002} + N_1\lambda_{101}^2 + N_2\lambda_{011}^2 + (N_3+8)\lambda_{002}^2 \\
\beta_{110} &= -\epsilon\lambda_{110} + (N_1+2)\lambda_{110}\lambda_{200} + (N_2+2)\lambda_{020}\lambda_{110} + N_3\lambda_{011}\lambda_{101} + 4\lambda_{110}^2 \\
\beta_{101} &= -\epsilon\lambda_{101} + (N_1+2)\lambda_{101}\lambda_{200} + N_2\lambda_{110}\lambda_{011} + (N_3+2)\lambda_{002}\lambda_{101} + 4\lambda_{101}^2 \\
\beta_{011} &= -\epsilon\lambda_{011} + N_1\lambda_{101}\lambda_{110} + (N_2+2)\lambda_{011}\lambda_{020} + (N_3+2)\lambda_{002}\lambda_{011} + 4\lambda_{011}^2
\end{aligned}$$

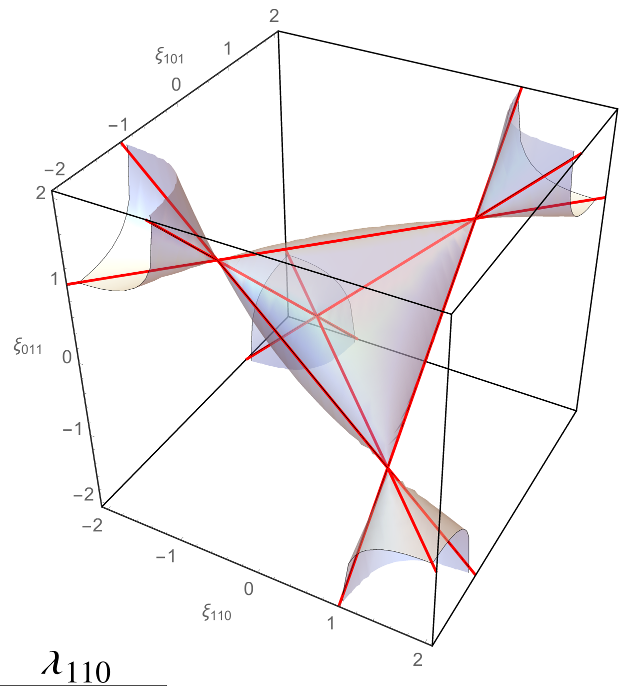
Note, at this order no $\eta = \mathcal{O}(\epsilon^2)$ effects!

Similar as in the two-field model, the critical hypersurface can be partitioned into different domains (see Figure). The complicated geometry allows for interesting flow topologies and FP dynamics as the number of field components N_i is varied.

In particular, we observe that FPs with continuous symmetry enhancement sit on the symmetry axes of the Cayley surface.

From the known scaling solutions of the two-field model it is straightforward to identify he following FPs:

- Isotropic FP (IFP)
- Decoupled FP (DFP)
- Partially decoupled biconal FP (DBFP)
- . . .



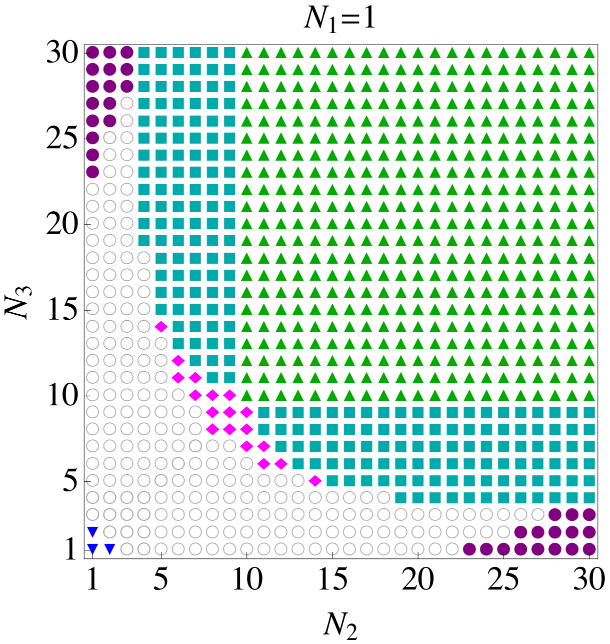
$$\xi_{110} = \frac{\lambda_{110}}{\sqrt{\lambda_{200}\lambda_{020}}},$$

$$\xi_{101} = \frac{\lambda_{101}}{\sqrt{\lambda_{200}\lambda_{002}}}, \quad \text{etc.}$$

Questions?

IR stability in three-field models

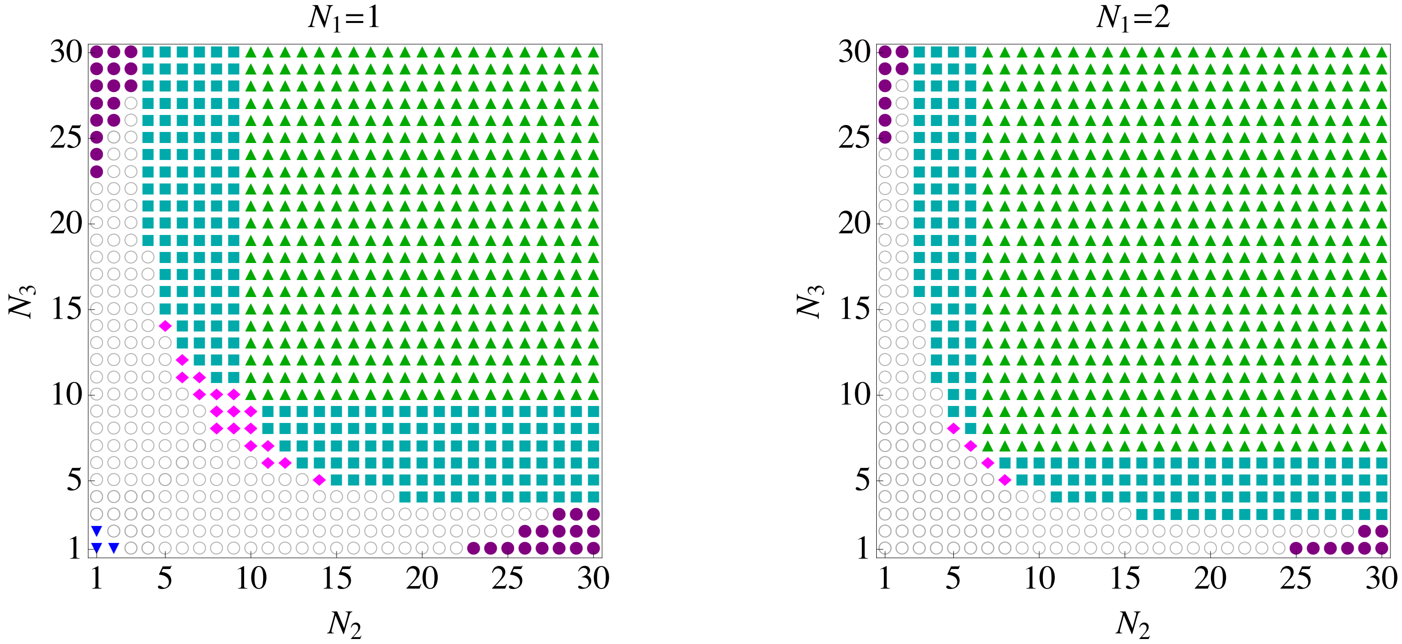
IR-stable FPs in (N_2, N_3) plane (one-loop ϵ-expansion, $\epsilon = 1$):



We observe the absence of IR-stable FP for small and intermediate values of N_i, in contrast to two-field model.

IR stability in three-field models

IR-stable FPs in (N_2, N_3) plane (one-loop ϵ-expansion, $\epsilon = 1$):



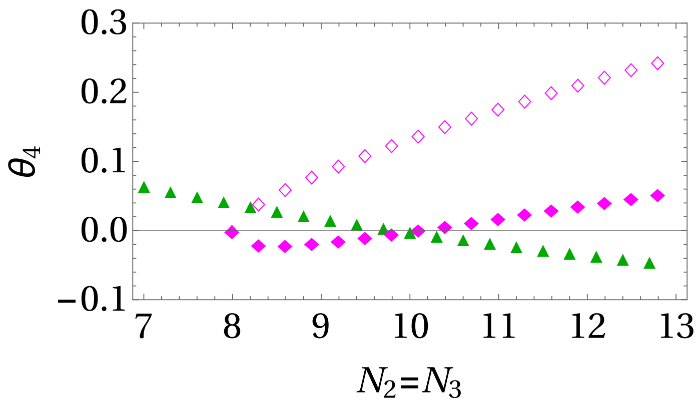
We observe the absence of IR-stable FP for small and intermediate values of N_i, in contrast to two-field model.

Emergence of new fully-coupled scaling solutions, while other FPs can be deduced from the two-field model. This pattern extends to general multi-field models.

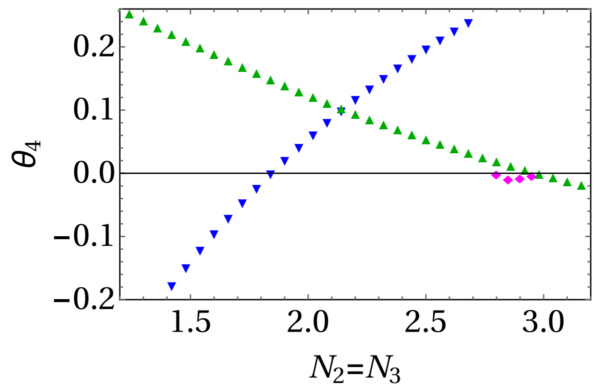
Questions?

Fully-coupled FPs in three-field models

Pair of fully-coupled FPs emerges from the complex plane – only one of them is IR-stable. This scenario does not appear in the two-field model!



One-loop ϵ-expansion



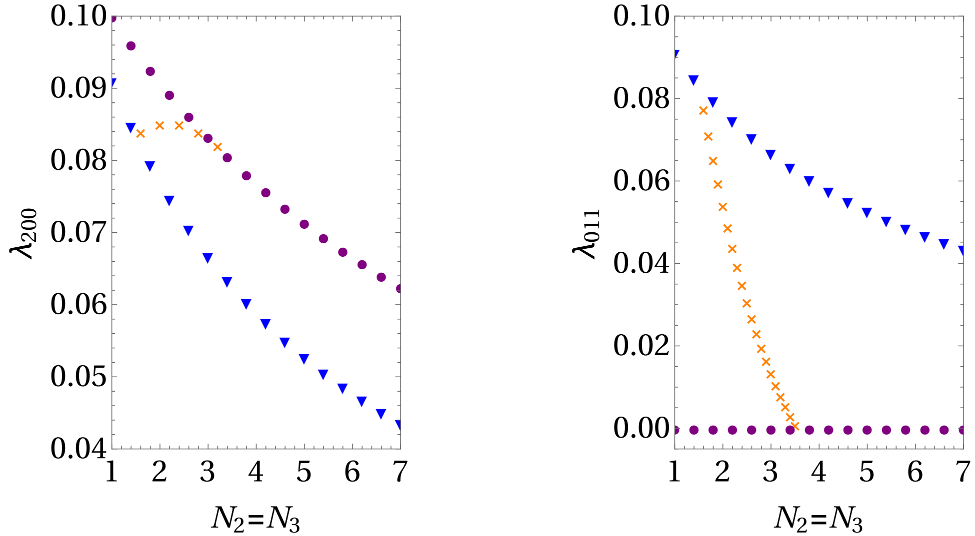
Functional RG
truncation to 4th order in the fields, $\eta \neq 0$

. . . only a small window of opportunity.

Exchange of IR-stability in three-field models

There are other examples of fully-coupled FPs that are never IR stable (without additional fine tuning). Some of these FPs feature the same type of stability trading that we observed already in the two-field model.

$N_1 = 1$



Summary

We have explored a large class of multicritical FPs in two-, three-, and four-field models. They come with an increasing number of (IR) relevant parameters that need to be tuned appropriately.

Even after the system has been tuned to the critical hypersurface, this does not guarantee that the IR-stable scaling solution is reached.

However, we observe that for values of N_i where a symmetry-enhanced fixed point exists and is IR stable all other fixed points feature a higher degree of fine-tuning required to reach them as IR FPs. Thus, symmetry-enhancement in the IR is the “natural” possibility.

It will be interesting to explore the coupling of similar scalar models to gravity.