Manifestly diffeomorphism invariant classical Exact Renormalization Group

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Contents

Introduction

- General Relativity (GR) is a perturbatively non-renormalizable theory.
- The field is the spacetime metric.
- Renormalization Group (RG) flow can be seen intuitively as describing physics at different scales of length by changing the resolution.
- Our work develops a manifestly diffeomorphism-invariant Exact RG.
- This approach avoids gauge fixing, giving results independent of coordinates.
- The diffeomorphism invariance also allows us to create a background-independent construction.
- The background-independent form does not make assumptions about the global structure of the spacetime.

Diffeomorphism transformations

Consider a general coordinate transformation

$$x'^{\mu} = x^{\mu} - \xi^{\mu}(x).$$

For a covariant derivative, *D*, metrics transform as

$$g'_{\mu\nu}(x') = \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} g_{\rho\sigma}(x) = g_{\mu\nu}(x) + 2g_{\lambda(\alpha}D_{\beta)}\xi^{\lambda} + \xi \cdot Dg_{\alpha\beta} \,.$$

So metric perturbations transform as

$$h_{\mu\nu}(x) \rightarrow h_{\mu\nu}(x) + \xi \cdot Dg_{\mu\nu} + 2g_{\lambda(\mu}D_{\nu)}\xi^{\lambda} \,.$$

A general covariant tensor transforms via the Lie derivative:

$$\pounds_{\xi} T_{\alpha_1 \cdots \alpha_n} = \xi^{\lambda} D_{\lambda} T_{\alpha_1 \cdots \alpha_n} + \sum_{i=1}^{n} T_{\alpha_1 \cdots \lambda \cdots \alpha_n} D_{\alpha_i} \xi^{\lambda},$$

We will later find it useful to generalize this further to objects with two position arguments.

Kadanoff blocking

In the Ising model, Kadanoff blocking is the process of grouping microscopic spins together to form macroscopic “blocked” spins, e.g. via a majority rule.

The continuous version integrates out the high-energy modes of a field to give a renormalized field, used in a renormalized action.

The blocking functional, *b*, is defined via the effective Boltzmann factor:

$$e^{-S[\varphi]} = \int \mathcal{D}\varphi_0 \; \delta\left[\varphi - b\left[\varphi_0\right]\right] e^{-S_{\mathrm{bare}}[\varphi_0]}.$$

There are an infinite number of possible Kadanoff blockings, but a simple linear example is

$$b[\varphi_0](x) = \int_y B(x-y)\varphi_0(y),$$

where the kernel, *B*, contains an infrared cutoff function.

The partition function must be invariant under change of cutoff scale, *Λ*, this will be ensured by construction, i.e.

$$\mathcal{Z} = \int \mathcal{D}\varphi \; e^{-S[\varphi]} = \int \mathcal{D}\varphi_0 \; e^{-S_{\mathrm{bare}}[\varphi_0]}.$$

Kadanoff blocking demands a suitable notion of locality that requires us to work exclusively in Euclidean signature.

RG Flow Equation

Differentiate the effective Boltzmann factor w.r.t. “RG time”:

$$\Lambda \frac{\partial}{\partial \Lambda} e^{-S[\varphi]} = -\int_x \frac{\delta}{\delta \varphi(x)} \int \mathcal{D}\varphi_0 \, \delta\left[\varphi - b\left[\varphi_0\right]\right] \Lambda \frac{\partial b[\varphi_0](x)}{\partial \Lambda} e^{-S_{\mathrm{bare}}[\varphi_0]}.$$

This can be rewritten in terms of the “rate of change of the blocking functional”, Ψ(x), as

$$\Lambda \frac{\partial}{\partial \Lambda} e^{-S[\varphi]} = \int_x \frac{\delta}{\delta \varphi(x)} \left(\Psi(x) e^{-S[\varphi]} \right).$$

This is now a general form for an RG flow equation for a single scalar field. If instead we choose a gauge field, we have (suppressing indices inside arguments)

$$\Lambda \frac{\partial}{\partial \Lambda} e^{-S[A]} = \int_x \frac{\delta}{\delta A_\mu(x)} \left(\Psi_\mu(x) e^{-S[A]} \right).$$

For gravity, we have

$$\Lambda \frac{\partial}{\partial \Lambda} e^{-S[g]} = \int_x \frac{\delta}{\delta g_{\mu\nu}(x)} \left(\Psi_{\mu\nu}(x) e^{-S[g]} \right).$$

Polchinski flow equation

We can specialize to the Polchinski flow equation for a single scalar field by setting

$$\Psi(x) = \frac{1}{2}\int_y \dot{\Delta}(x,y)\,\frac{\delta\Sigma}{\delta\varphi(y)}.$$

The effective propagator, $\Delta = c(p^2/\Lambda^2)p^{-2}$, contains an ultraviolet cutoff profile, *c*.

The seed action, $\hat{S}$, which appears in $\Sigma = S - 2\hat{S}$,

is the regularized kinetic term (in the Polchinski flow equation). We can freely generalize to allow 3-point and higher additions.

$$\hat{S} = \frac{1}{2}\int_x \partial_\mu\varphi\; c^{-1}\left(-\frac{\partial^2}{\Lambda^2}\right)\partial_\mu\varphi + \cdots$$

This freedom comes from the infinite number of possible Kadanoff blockings. The Polchinski flow equation can be neatly expressed as

$$\dot{S} = \frac{1}{2}\frac{\delta S}{\delta\varphi}\cdot\dot{\Delta}\cdot\frac{\delta\Sigma}{\delta\varphi} - \frac{1}{2}\frac{\delta}{\delta\varphi}\cdot\dot{\Delta}\cdot\frac{\delta\Sigma}{\delta\varphi},$$

where $f\cdot W\cdot g := \int_x f(x)W\left(-\frac{\partial^2}{\Lambda^2}\right)g(x).$

Gauge invariant flow equation

The generalization of the Polchinski flow equation to a single gauge field looks very similar:

$$\dot{S} = \frac{1}{2}\frac{\delta S}{\delta A_\mu}\cdot\{\dot{\Delta}\}\cdot\frac{\delta\Sigma}{\delta A_\mu} - \frac{1}{2}\frac{\delta}{\delta A_\mu}\cdot\{\dot{\Delta}\}\cdot\frac{\delta\Sigma}{\delta A_\mu}.$$

The crucial difference to the scalar case is that the kernel must be covariantized to maintain manifest gauge invariance.

There are an infinite number of ways to do this, a simple way is to replace the partial derivatives with covariant derivatives,

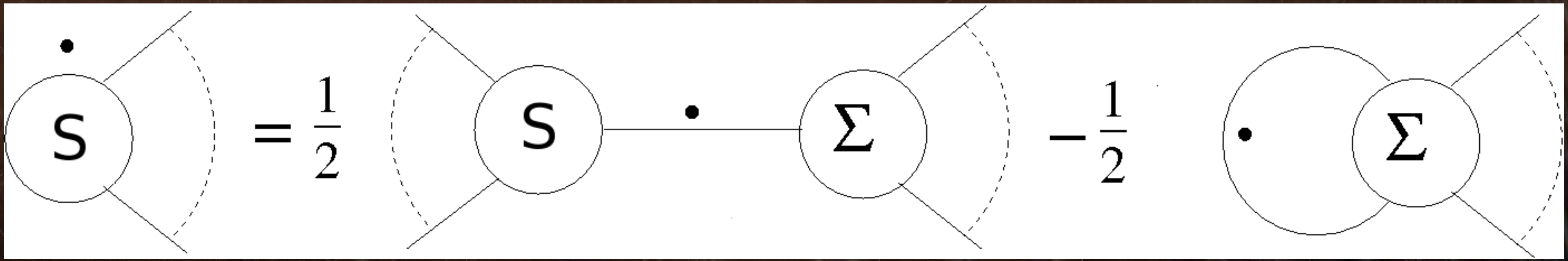
$D_\mu := \partial_\mu - iA_\mu,$ so the kernel now has an expansion in $-\frac{D^2}{\Lambda^2}.$

We continue to call Δ the "effective propagator", although instead of inverting the tree-level 2-point function, it maps it onto the transverse projector:

$$\Delta S_{0\mu\nu}^{\ AA} = \delta_{\mu\nu} - p_\mu p_\nu/p^2 .$$

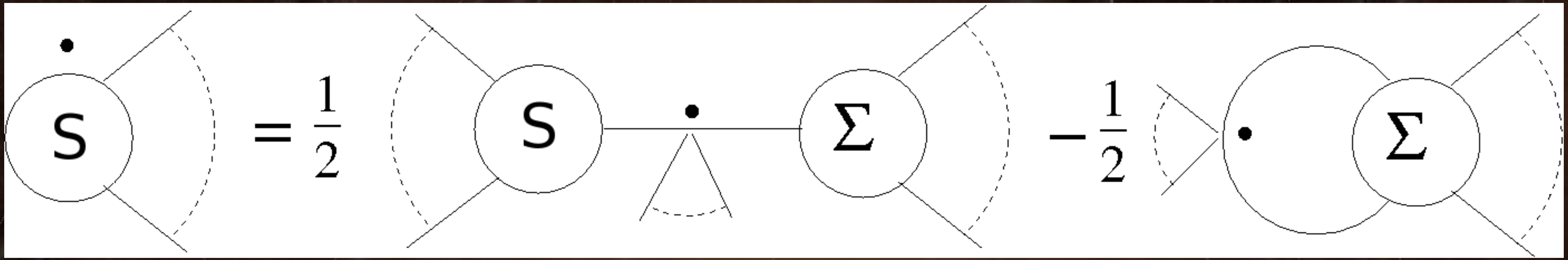
Diagrammatic view

We can visualize the flow equation expansion for scalar field theory diagrammatically:



$$\dot{S} = \frac{1}{2}\frac{\delta S}{\delta \varphi} \cdot \dot{\Delta} \cdot \frac{\delta \Sigma}{\delta \varphi} - \frac{1}{2}\frac{\delta}{\delta \varphi} \cdot \dot{\Delta} \cdot \frac{\delta \Sigma}{\delta \varphi}$$

In gauge and gravity theories, the kernel also has an expansion due to covariantization:



Loop expansion of the action

To preserve manifest gauge invariance, wavefunction renormalization must be avoided. For gauge theories, this is achieved by writing the coupling, *g*, as an overall scaling factor:

$$S[A](g) = \frac{1}{4g^2} tr \int_x F_{\mu\nu} c^{-1} \left(-\frac{D^2}{\Lambda^2} \right) F_{\mu\nu} + \cdots$$

The effective action is then written as a loopwise expansion that is also an expansion in powers of *g:*

$$S = \frac{1}{g^2} S_0 + S_1 + g^2 S_2 + \cdots \quad \text{and} \quad \Sigma_g = g^2 S - 2\hat{S}.$$

The left hand side of the flow equation then reads as

$$\dot{S} = \frac{1}{g^2} S_0 + S_1 + g^2 S_2 + \cdots + \beta \left(-\frac{1}{g^3} S_0 + g S_2 + \cdots \right)$$

The β-functions have an expansion: $\beta := \Lambda \partial_\Lambda g = \beta_1 g^3 + \beta_2 g^5 + \cdots$

Comment on wavefunction renormalization

To preserve manifest gauge invariance, we require that the gauge transformation of the field keeps the form

$$\delta A_\mu = [D_\mu, \omega(x)].$$

If wavefunction renormalization were required, the renormalized field would scale as

$$A_\mu^R = Z^{-1/2} A_\mu.$$

The gauge transformation would then change to

$$\delta A_\mu^R = Z^{-1/2} \partial_\mu \omega - i[A_\mu^R, \omega].$$

This preserves the relation only if we require that $Z = 1$

and $A_\mu = A_\mu^R.$

This differs from the usual method, which instead introduces BRST ghost fields and replaces manifest gauge invariance with BRST invariance.

Additional regularization for loops

The manifestly gauge invariant ERG for *SU(N)* requires additional regularization at 1-loop level using covariant Pauli-Villars fields. The elegant way to do this is with *SU(N|N)* regularization.

The field is promoted to a supermatrix of bosonic components, *A*, and fermionic components, *B*:

$$\mathcal{A}_\mu = \begin{pmatrix} A^1_\mu & B_\mu \\ \bar{B}_\mu & A^2_\mu \end{pmatrix} + \mathcal{A}^0_\mu \mathcal{I}, \quad \text{and} \quad D_\mu = \partial_\mu - i\mathcal{A}_\mu.$$

The action is built in a similar way:

$$S = \frac{1}{4g^2} str \int \mathcal{F}_{\mu\nu} c^{-1}\left(-\frac{D^2}{\Lambda^2}\right)\mathcal{F}_{\mu\nu} + \cdots$$

where $str \begin{pmatrix} X^{11} & X^{12} \\ X^{21} & X^{22} \end{pmatrix} = tr_1 X^{11} - tr_2 X^{22}.$

This supersymmetry is spontaneously broken by a super-Higgs mechaism with a mass at order *Λ*, so that the physical *SU(N)* can be recovered at low energy.

A similar procedure would be required for the 1-loop manifestly diffeomorphism invariant ERG for gravity.

Diffeomorphism invariant flow equation

The background-independent generalization of the Polchinski flow equation to gravity is

$$\dot{S} = \int_x \frac{\delta S}{\delta g_{\mu\nu}(x)} \int_y K_{\mu\nu\rho\sigma}(x,y) \frac{\delta \Sigma}{\delta g_{\rho\sigma}(y)} - \int_x \frac{\delta}{\delta g_{\mu\nu}(x)} \int_y K_{\mu\nu\rho\sigma}(x,y) \frac{\delta \Sigma}{\delta g_{\rho\sigma}(y)},$$

The kernel, which transforms as a two-argument generalization of a tensor, is

$$K_{\mu\nu\rho\sigma}(x,y) = \frac{1}{\sqrt{g}} \delta(x-y) \left(g_{\mu(\rho} g_{\sigma)\nu} + j g_{\mu\nu} g_{\rho\sigma} \right) \dot{\Delta}.$$

The de Witt supermetric parameter, *j*, determines how modes propagate in the flow equation. The value of *j* is determined by our choice of scheme.

The "effective propagator", *Δ*, in the fixed-background description is given by

$$\Delta = \frac{c(p^2/\Lambda^2)}{p^d},$$

where *d* is the dimension of the *Λ*-independent part of the 2-point function. Background-independently, we only have *Δ*'s RG time derivative, which is a local expansion in covariant derivatives.

Flow equation at the classical level

The flow equation can be split into two index structures, which are the "cross-contracted" and "two-traces" forms. At the classical level, we have

$$\dot{S}|_{c.c.} = \int_x \frac{\delta S}{\delta g_{\mu\nu}} \frac{g_{\mu(\rho}g_{\sigma)\nu}}{\sqrt{g}} \dot{\Delta} \frac{\delta \Sigma}{\delta g_{\rho\sigma}}, \quad \text{and}$$

$$\dot{S}|_{t.t.} = \int_x \frac{\delta S}{\delta g_{\mu\nu}} \frac{g_{\mu\nu}g_{\rho\sigma}}{\sqrt{g}} \dot{\Delta} \frac{\delta \Sigma}{\delta g_{\rho\sigma}}.$$

The full flow equation is a linear combination of both structures:

$$\dot{S} = \dot{S}|_{c.c.} + j\dot{S}|_{t.t.}.$$

We can write the classical flow equation in compact form as

$$\dot{S} = -a_0[S, \Sigma],$$

where a_0 is a symmetric bilinear map between actions. We could also use a_0, a symmetric bilinear map between Lagrangian terms:

$$\dot{\mathcal{L}} = -\mathsf{a}_0[\mathcal{L}, \mathcal{L} - 2\hat{\mathcal{L}}]\,.$$

As with scalar and gauge cases, we only require the seed action to match the fixed-point action up to the 2-point level, which for gravity means up to quadratic terms in Riemann tensor.

Brief comments on *j*

The value of j determines the balance of modes in the flow equation, take for example

$$j \longrightarrow \infty,$$

which is the case where the kernel only keeps the index structure that traces both sides. This choice ensures that only the conformal mode propagates in the flow equation. To see this, let's bring the conformal factor outside the metric:

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} e^{\sigma}\,.$$

We find that we can rewrite the flow equation as merely a flow equation for the conformal factor:

$$\frac{\delta S}{\delta \sigma} = g_{\mu\nu} \frac{\delta S}{\delta g_{\mu\nu}},$$

$$\dot{S} \sim \int_x \frac{\delta S}{\delta \sigma} \frac{\dot{\Delta}}{\sqrt{g}} \frac{\delta \Sigma}{\delta \sigma} - \int_x \frac{\delta}{\delta \sigma} \frac{\dot{\Delta}}{\sqrt{g}} \frac{\delta \Sigma}{\delta \sigma}.$$

Conversely, in *D* dimensions, *j* = -1/*D* prevents the conformal mode from propagating in the flow equation. We specialise to *D* = 4 in this work.

Background-independent expansion of the action

The Lagrangian can be arranged as an expansion in local diffeomorphism-invariant scalar operators of even mass dimension, d = 2*i*, which corresponds to the number of derivatives:

$$\mathcal{L} = \sum_{i=0}^{\infty} \sum_{\alpha_i} g_{2i,\alpha_i} \, \mathcal{O}_{2i,\alpha_i} .$$

For a Lagrangian of dimension *l*, the couplings have mass dimension *l-d*. The kernel also gives as a series expansion in derivatives:

$$\dot{\Delta}(-\nabla^2) = \sum_{k=0}^{\infty} \frac{1}{k!} \dot{\Delta}^{(k)}(0) \left(-\nabla^2\right)^k .$$

Because of this, our bilinear mapping of two operators also follows a series expansion:

$$\mathsf{a}_0[\mathcal{O}_{d_1}, \mathcal{O}_{d_2}] = \sum_{k=0}^{\infty} \mathsf{a}_0^k[\mathcal{O}_{d_1}, \mathcal{O}_{d_2}].$$

We can integrate these terms w.r.t. RG time to reproduce the operator expansion of the action. The higher dimension operators are suppressed by increasingly negative powers of *Λ*. Dimensionless coupling terms are reproduced as “integration constants”.

Simple examples

The only zero-dimensional operator is the effective cosmological constant. The flow equation at this order is simply a first order differential equation:

$$\dot{g}_0 = g_0(2\hat{g}_0 - g_0)\, \mathsf{a}_0^0[1,1]\,.$$

The only 2-dimensional operator is the Einstein-Hilbert term, which is similarly straightforward:

$$\dot{g}_2 = 2(g_0\hat{g}_2 + \hat{g}_0 g_2 - g_0 g_2)\, \frac{\mathsf{a}_0^0[\mathcal{O}_2, 1]}{\mathcal{O}_2}\,.$$

For evaluating bilinear operations with a cosmological constant term, it is helpful to note that

$$\mathsf{a}_0^k[\mathcal{O}_d, 1] = 0 \qquad \forall k > 0.$$

Consider $a_0\left[S, \int_x \sqrt{g}\right] = -\frac{1}{4}(1+4j)\dot{\Delta}(0)\int_x g_{\mu\nu}\frac{\delta S}{\delta g_{\mu\nu}}\,.$

Remembering that $\frac{\delta S}{\delta \sigma} = g_{\mu\nu}\frac{\delta S}{\delta g_{\mu\nu}}$, one can infer that the last factor counts metric powers:

$$\mathsf{a}_0[\mathcal{O}_d, 1] = \mathsf{a}_0^0[\mathcal{O}_d, 1] = \frac{1}{8}(d-4)(1+4j)\dot{\Delta}(0)\,\mathcal{O}_d\,.$$

The “Weyl scheme”

The seed action matches the fixed-point action up to curvature-squared terms and determines the form of the effective propagator.

Let us construct a classical fixed-point action using the following seed Lagrangian:

$$\hat{\mathcal{L}} = 2R_{\alpha\beta}\, c^{-1}(-\nabla^2/\Lambda^2)\, R^{\alpha\beta} + 2sR\, c^{-1}(-\nabla^2/\Lambda^2)\, R.$$

The flow equation yields an expansion that goes like

$$\dot{\mathcal{L}} = 4\, \mathsf{a}_0[R_{\mu\nu}R^{\mu\nu}, R_{\alpha\beta}R^{\alpha\beta}] + 8s\, \mathsf{a}_0[R_{\mu\nu}R^{\mu\nu}, R^2] + 4s^2 \mathsf{a}_0[R^2, R^2] + \cdots,$$

Requiring quasi-locality constrains the kernel expansion to give us

$$\int \frac{d\Lambda}{\Lambda}\dot{\Delta} = \frac{1}{2\Lambda^4}\, c''(0) - \frac{1}{6\Lambda^6}\, c'''(0)\nabla^2 + O(\nabla^4)\,.$$

There are no 6-dimensional operators in *S*. Using the flow equation expansion above, solving at the curvature-squared level for *d* = 8, 10 gives us

$$1 + 4j + 4s(2+3s)(1+3j) = s.$$

This fixes the value of *j* to be $j = -\frac{1}{4}\,\frac{1+4s}{1+3s}.$

The “Einstein scheme”

Now consider instead an action that begins with an Einstein-Hilbert term

$$S = \frac{1}{32\pi G}\int_x \sqrt{g}\left(-2R + \frac{2}{\Lambda^2}R_{\mu\nu}d\left(\frac{-\nabla^2}{\Lambda^2}\right)R^{\mu\nu} + \frac{2j}{\Lambda^2}Rd\left(\frac{-\nabla^2}{\Lambda^2}\right)R + \cdots\right),$$

Where *d* is related to the inverted cutoff. As with gauge theories, we will want to rescale the action to factor out the coupling. This gives us the following seed Lagrangian:

$$\hat{\mathcal{L}} = -2R + \frac{2}{\Lambda^2}R_{\mu\nu}\,d(-\nabla^2/\Lambda^2)R^{\mu\nu} + \frac{2}{\Lambda^2}jR\,d(-\nabla^2/\Lambda^2)R\,.$$

The quadratic (and higher) terms in the Riemann tensor can be derived from the flow equation:

$$g_{4,1}R^2 + g_{4,2}R^{\mu\nu}R_{\mu\nu} = 4\int\frac{d\Lambda}{\Lambda}\,\mathsf{a}_0^0[R,R] = -2\frac{c'(0)}{\Lambda^2}\left(R_{\mu\nu}R^{\mu\nu} + jR^2\right).$$

We can maintain this structure at all orders of the curvature-squared level provided that *j* = -1/2 or -1/3.

Loop expansion

Let us consider the loop expansion of the action:

$$S = \frac{1}{\tilde{\kappa}} S_0 + S_1 + \tilde{\kappa} S_2 + \tilde{\kappa}^2 S_3 \cdots \quad \text{and} \quad \Sigma_{\tilde{\kappa}} = \tilde{\kappa} S - 2\hat{S}.$$

For the Einstein scheme, the coupling is related to Newton's constant: $\tilde{\kappa} = 32\pi G$

Remembering that the flow equation is written as $\dot{S} = -a_0[S, \Sigma] + a_1[\Sigma],$

we can expand it out loopwise to get

$$\frac{1}{\tilde{\kappa}} \dot{S}_0 + \dot{S}_1 + \tilde{\kappa} \dot{S}_2 + \tilde{\kappa}^2 \dot{S}_3 + \cdots + \beta \left(-\frac{1}{\tilde{\kappa}^2} S_0 + S_2 + 2\tilde{\kappa} S_3 + \cdots \right) = -\frac{1}{\tilde{\kappa}} a_0[S_0, S_0 - 2\hat{S}]$$
$$-2a_0[S_0 - \hat{S}, S_1] + a_1[S_0 - 2\hat{S}] + \tilde{\kappa} \left(-2a_0[S_0 - \hat{S}, S_2] - a_0[S_1, S_1] + a_1[S_1] \right) + \cdots .$$

The classical equation is recovered for $\tilde{\kappa} \longrightarrow 0.$

The loop expansion of the beta function goes like

$$\beta := \Lambda \partial_\Lambda \tilde{\kappa} = \beta_1 \Lambda^2 \tilde{\kappa}^2 + \beta_2 \Lambda^4 \tilde{\kappa}^3 + \cdots .$$

Fixed-background form

If we fix a Euclidean background metric, we can define the graviton field as the perturbation to that background:

$$h_{\mu\nu}(x) := g_{\mu\nu}(x) - \delta_{\mu\nu}.$$

The position representation is related to a momentum representation via a Fourier transform:

$$h_{\mu\nu}(x) = \int đp\, \mathrm{e}^{-ip\cdot x} h_{\mu\nu}(p), \quad \text{where} \quad đp := \frac{d^D p}{(2\pi)^D}.$$

The action is defined as a series expansion in the perturbation:

$$\begin{aligned} S \;&=\; \int đp\, \delta(p) \mathcal{S}^{\mu\nu}(p) h_{\mu\nu}(p) + \frac{1}{2}\int đp\, đq\, \delta(p+q) \mathcal{S}^{\mu\nu\rho\sigma}(p,q) h_{\mu\nu}(p) h_{\rho\sigma}(q) \\ &\quad + \frac{1}{3!}\int đp\, đq\, đr\, \delta(p+q+r) \mathcal{S}^{\mu\nu\rho\sigma\alpha\beta}(p,q,r) h_{\mu\nu}(p) h_{\rho\sigma}(q) h_{\alpha\beta}(r) + \cdots \end{aligned}$$

In this form, we are able to define an “effective propagator”, $\Delta := \dfrac{1}{p^l} c\left(\dfrac{p^2}{\Lambda^2}\right).$

Where *l* = 2 in the Einstein scheme and 4 in the Weyl scheme.

Expansion of the metric determinant

The action has a factor of $\sqrt{g}$. The kernel has a factor of $1/\sqrt{g}$.

We can use the momentum-independent Ward identities to determine the expansion of these factors in metric perturbations, or we can do it more directly:

$$\sqrt{\det(g_{\mu\nu})}^{\,l} = e^{\frac{l}{2}(\ln(\delta_{\mu\nu}+h_{\mu\nu}))},$$

$$\sqrt{g}^{l} = 1 + l\frac{h}{2} - l\frac{h_{\mu\nu}h^{\mu\nu}}{4} + l^2\frac{h^2}{8} + l\frac{h_{\mu\nu}h^{\mu\rho}h^{\nu}{}_{\rho}}{6} - l^2\frac{h_{\mu\nu}h^{\mu\nu}h}{8} + l^3\frac{h^3}{48} + \cdots$$

A "cosmological constant" term would enter the action simply as $\int_x \sqrt{g} \times \text{constant}$

Thus a "cosmological constant" term would introduce a corresponding 1-point function because we are expanding around a Euclidean background rather than the more natural de Sitter background.

n-point expansion of the kernel

We wish to calculate the *n*-point functions of the kernel, which is contains an expansion in covariant derivatives.

To get the linear part in metric perturbations, we first calculate the result, *H*, of using just the linear term in the derivative expansion:

$$(-\nabla^2)(p,r)T^{\rho\sigma}(-r) = H^{\alpha\beta}{}_{\gamma\delta}{}^{\rho\sigma}(p,r)T^{\gamma\delta}(-r)h_{\alpha\beta}(p)\,.$$

We then generalize to any order in the derivative expansion:

$$(-\nabla^2)^{n+1}(p,r)T^{\rho\sigma}(-r) = \sum_{i=0}^{n}([p-r]^2)^{n-i}H^{\alpha\beta}{}_{\gamma\delta}{}^{\rho\sigma}(p,r)(r^2)^i T^{\gamma\delta}(-r)h_{\alpha\beta}(p)\,.$$

The sum is over terms in a geometric series, which goes like

$$\sum_{i=0}^{n}([p-r]^2)^{n-i}(r^2)^i = \frac{(p-r)^{2(n+1)} - r^{2(n+1)}}{(p-r)^2 - r^2}.$$

Finally, we add up all orders of the expansion

$$\dot{\Delta}(-\nabla^2)(p,r)T^{\rho\sigma}(-r) = \frac{\dot{\Delta}\left(|p-r|^2\right) - \dot{\Delta}(r^2)}{|p-r|^2 - r^2}H^{\alpha\beta}{}_{\gamma\delta}{}^{\rho\sigma}(p,r)T^{\gamma\delta}(-r)h_{\alpha\beta}(p).$$

Transverse 2-point structures

The kinetic term gives us transverse 2-point functions. We wish to use diffeomorphism invariance to constrain what 2-point functions we have. We start with the most general structure with two derivatives:

$$S^{(2)}_{2-\text{momenta}} = \int đp \, \left(a_1 h p^2 h + a_2 h_{\alpha\beta} p^2 h^{\alpha\beta} + a_3 h p_\alpha p_\beta h^{\alpha\beta} + a_4 h^{\alpha\beta} p_\alpha p_\gamma h_\beta{}^\gamma \right).$$

We require the linearized diffeomorphism transformation to be zero:

$$\begin{aligned} 0 \quad = \quad & 4a_1 h p^2 p\cdot\xi + 4a_2 h^{\alpha\beta} p^2 p_\alpha \xi_\beta + 2a_3 h p^2 p\cdot\xi \\ & + 2a_3 h^{\alpha\beta} p_\alpha p_\beta p\cdot\xi + 2a_4 h^{\alpha\beta} p^2 p_\alpha \xi_\beta + 2a_4 h^{\alpha\beta} p_\alpha p_\beta p\cdot\xi. \end{aligned}$$

This gives us one unique structure, which corresponds to the Einstein-Hilbert action:

$$a_1 = -a_2 = -a_3/2 = a_4/2.$$

But what if we allow for four or more derivatives? The most general structure with four derivatives is

$$\begin{aligned} S^{(2)}_{4-\text{momenta}} \quad = \quad & \int đp \, \left(b_1 h^{\alpha\beta} p^4 h_{\alpha\beta} + b_2 h p^4 h + b_3 h^{\alpha\beta} p^2 p_\alpha p_\beta h \right. \\ & \left. + b_4 h^{\alpha\beta} p^2 p_\alpha p_\gamma h_\beta{}^\gamma + b_5 h^{\alpha\beta} p_\alpha p_\beta p_\gamma p_\delta h^{\gamma\delta} \right). \end{aligned}$$

Transverse 2-point functions

Again, we set the linearized diffeomorphism transformation to zero:

$$\begin{aligned} 0 \;\; = \;\; & 4b_1 h^{\alpha\beta} p^4 p_\alpha \xi_\beta + 4b_2 h p^4 p\cdot\xi + 2b_3 h^{\alpha\beta} p^2 p_\alpha p_\beta p\cdot\xi \\ & 2b_3 h p^4 p\cdot\xi + 2b_4 h^{\alpha\beta} p^4 p_\alpha \xi_\beta + 2b_4 h^{\alpha\beta} p^2 p_\alpha p_\beta p\cdot\xi \\ & + 4b_5 h^{\alpha\beta} p^2 p_\alpha p_\beta p\cdot\xi. \end{aligned}$$

This requires that $b_5 = b_1 + b_2, b_4 = -2b_1, \text{and } b_3 = -2b_2,$

giving us two linearly independent structures

$$2S_a^{(2)} = h^{\mu\nu} p^4 h_{\mu\nu} - 2h^{\mu\nu} p^2 p_\mu p_\rho h_\nu{}^\rho + h^{\mu\nu} p_\mu p_\nu p_\rho p_\sigma h^{\rho\sigma},$$

$$2S_b^{(2)} = h p^4 h - 2h^{\mu\nu} p^2 p_\mu p_\nu h + h^{\mu\nu} p_\mu p_\nu p_\rho p_\sigma h^{\rho\sigma}.$$

The two transverse 2-point functions can be written in a factorized form:

$$\mathcal{S}_a^{\mu\nu\rho\sigma}(-p,p) = \left(p^2 \delta^{(\mu|(\rho} - p^{(\mu|} p^{(\rho}\right)\left(p^2 \delta^{\sigma)|\nu)} - p^{\sigma)} p^{|\nu)}\right),$$

$$\mathcal{S}_b^{\mu\nu\rho\sigma}(-p,p) = \left(p^2 \delta^{\mu\nu} - p^\mu p^\nu\right)\left(p^2 \delta^{\rho\sigma} - p^\rho p^\sigma\right).$$

Interpreting the 2-point functions

Increasing the number of derivatives further does not give us any new structures.

Consider a structure: $$S^{(2)} = aS_a^{(2)} + bS_b^{(2)}.$$

$a = 2,\ b = 0$ corresponds to $\int_x \sqrt{g} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma},$

$a = 0,\ b = 2$ corresponds to $\int_x \sqrt{g} R^2,$

$a = b = \frac{1}{2}$ corresponds to $\int_x \sqrt{g} R_{\mu\nu} R^{\mu\nu},$

$a = -b = \frac{1}{2p^2}$ corresponds to $-\int_x \sqrt{g} R.$

The quadratic terms are related by the Gauss-Bonnet topological invariant:

$$\frac{1}{32\pi^2} \int_M \left(R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right).$$

Ward identities for the action

Recall the diffeomorphism transformation for a metric perturbation in position representation:

$$\delta h_{\alpha\beta} = 2(\delta + h)_{\lambda(\alpha}\partial_{\beta)}\xi^{\lambda} + \xi \cdot \partial h_{\alpha\beta}\,.$$

We can write the variation in momentum representation as

$$i\delta h_{\alpha\beta}(p) = 2p_{(\alpha}\xi_{\beta)} + \int đp'\, đk'\, \delta(p - p' - k') \Big(2p'_{(\alpha}h_{\beta)\lambda}(k')\xi^{\lambda}(p') + \xi(p') \cdot k' h_{\alpha\beta}(k')\Big)\,.$$

The Ward identities follow from requiring diffeomorphism invariance of the action:

$$\begin{aligned} 2p_{1\mu_1}\mathcal{S}^{\mu_1\nu_1\cdots\mu_n\nu_n}(p_1,\cdots,p_n) \quad = \quad & -\sum_{i=2}^{n}\pi_{2i}\Big\{ p_2^{\nu_1}\mathcal{S}^{\mu_2\nu_2\cdots\mu_n\nu_n}(p_1+p_2,p_3,\cdots,p_n) \\ & -2p_{1\alpha}\delta^{\nu_1(\nu_2}\mathcal{S}^{\mu_2)\alpha\mu_3\nu_3\cdots\mu_n\nu_n}(p_1+p_2,p_3,\cdots,p_n)\Big\}, \end{aligned}$$

where momentum conservation is assumed: $\displaystyle\sum_{i=1}^{n} p_i = 0,$

and π_{2i} is the transposition operator that swaps the labels “2” and “i” in all Lorentz indices and momenta

Differential Ward identities

For the case where one momentum is zero (tadpoles), the Ward identities directly give us the form of the (*n+1*)-point functions in terms of the *n*-point functions. We set one of the momenta to be vanishing:

$$(p_0, p_1, p_2, \cdots, p_n) = (\epsilon, p_1 - \epsilon, p_2, \cdots, p_n).$$

Differentiating the Ward identity with respect to the vanishing momentum and applying momentum conservation, we get:

$$\begin{aligned}-2\mathcal{S}^{\alpha\beta\mu_1\nu_1\cdots\mu_n\nu_n}(0, p_1, \cdots, p_n) \quad &= \left(\sum_{i=1}^{n} p_i^\beta \partial_i^\alpha - \delta^{\alpha\beta}\right)\mathcal{S}^{\mu_1\nu_1\cdots\mu_n\nu_n}(p_1, \cdots, p_n) \\ &+2\sum_{i=1}^{n}\pi_{1i}\,\delta^{\beta(\nu_1}\mathcal{S}^{\mu_1)\alpha\mu_2\nu_2\cdots\mu_n\nu_n}(p_1, \cdots, p_n).\end{aligned}$$

Tending all momenta to zero gives us the Ward identities for momentum-independent parts of the action:

$$2\mathcal{S}^{\mu_1\nu_1\cdots\mu_n\nu_n}(\underline{0}) = \delta^{\mu_1\nu_1}\mathcal{S}^{\mu_2\nu_2\cdots\mu_n\nu_n}(\underline{0}) - 2\sum_{i=2}^{n}\pi_{2i}\,\delta^{\nu_1(\nu_2}\mathcal{S}^{\mu_2)\mu_1\mu_3\nu_3\cdots\mu_n\nu_n}(\underline{0})\,.$$

Deriving Ward identities for the kernel

The kernel diffeomorphism transforms as the two-argument generalization of a tensor:

$$\begin{aligned}\pounds_\xi K_{\mu\nu\rho\sigma}(x,y) &= \xi(x)\cdot\partial_x K_{\mu\nu\rho\sigma}(x,y) + \xi(y)\cdot\partial_y K_{\mu\nu\rho\sigma}(x,y)\\ &\quad +2K_{\lambda(\mu|\rho\sigma}(x,y)\partial_{x|\nu)}\xi^\lambda(x) + 2K_{\mu\nu\lambda(\rho|}(x,y)\partial_{y|\sigma)}\xi^\lambda(y),\end{aligned}$$

The kernel transforms in momentum representation like

$$\begin{aligned}i\delta K_{\mu\nu\rho\sigma}(q,r) &= -\xi(p')\cdot(p'+q)K_{\mu\nu\rho\sigma}(p'+q,r) - \xi(p')\cdot(p'+r)K_{\mu\nu\rho\sigma}(q,p'+r)\\ &\quad +2\xi^\lambda(p')p'_{(\mu}K_{\nu)\lambda\rho\sigma}(p'+q,r) + 2\xi^\lambda(p')p'_{(\rho|}K_{\mu\nu|\sigma)\lambda}(q,p'+r).\end{aligned}$$

The kernel can be written in momentum representation as an expansion in metric perturbations:

$$K_{\mu\nu\rho\sigma}(q,r) = \mathcal{K}_{\mu\nu\rho\sigma}(q,r) + \int đp_1\;\delta(p_1+q+r)\mathcal{K}^{\alpha_1\beta_1}_{\mu\nu\rho\sigma}(p_1,q,r)h_{\alpha_1\beta_1}(p_1) + \cdots$$

Taking into account also the transformation of the metric perturbations, we can derive an overall set of Ward identities for the kernel...

Ward identities for the kernel

The result is
$$
\begin{aligned}
&2p'_\gamma \mathcal{K}^{\gamma\delta\alpha_1\beta_1\cdots\alpha_n\beta_n}{}_{\mu\nu\rho\sigma}(p',p_1,\cdots,p_n,q,r) = \\
&-(p'+q)^\delta \mathcal{K}^{\alpha_1\beta_1\cdots\alpha_n\beta_n}{}_{\mu\nu\rho\sigma}(p_1,\cdots,p_n,q+p',r) \\
&-(p'+r)^\delta \mathcal{K}^{\alpha_1\beta_1\cdots\alpha_n\beta_n}{}_{\mu\nu\rho\sigma}(p_1,\cdots,p_n,q,r+p') \\
&+2\delta^{\lambda\delta} p'_{(\mu|} \mathcal{K}^{\alpha_1\beta_1\cdots\alpha_n\beta_n}{}_{|\nu)\lambda\rho\sigma}(p_1,\cdots,p_n,q+p',r) \\
&+2\delta^{\lambda\delta} p'_{(\rho|} \mathcal{K}^{\alpha_1\beta_1\cdots\alpha_n\beta_n}{}_{\mu\nu|\sigma)\lambda}(p_1,\cdots,p_n,q,r+p') \\
&-\sum_{i=1}^{n} \pi_{i1} \left\{ p_1^\delta \mathcal{K}^{\alpha_1\beta_1\cdots\alpha_n\beta_n}{}_{\mu\nu\rho\sigma}(p'+p_1,p_2,\cdots,p_n,q,r) \right. \\
&\left. +2p'_\lambda \delta^{\delta(\alpha_1}\mathcal{K}^{\beta_1)\lambda\alpha_2\beta_2\cdots\alpha_n\beta_n}{}_{\mu\nu\rho\sigma}(p'+p_1,p_2,\cdots,p_n,q,r) \right\}.
\end{aligned}
$$

The momentum-independent part satisfies
$$
\begin{aligned}
\mathcal{K}^{\gamma\delta\alpha_1\beta_1\cdots\alpha_n\beta_n}{}_{\mu\nu\rho\sigma}(\underline{0}) \quad = \quad & -\frac{1}{2}\delta^{\gamma\delta}\mathcal{K}^{\alpha_1\beta_1\cdots\alpha_n\beta_n}{}_{\mu\nu\rho\sigma}(\underline{0}) \\
& +\delta^{\lambda(\gamma}\delta^{\delta)}{}_{(\mu|}\mathcal{K}^{\alpha_1\beta_1\cdots\alpha_n\beta_n}{}_{|\nu)\lambda\rho\sigma}(\underline{0}) \\
& +\delta^{\lambda(\gamma}\delta^{\delta)}{}_{(\rho|}\mathcal{K}^{\alpha_1\beta_1\cdots\alpha_n\beta_n}{}_{\mu\nu|\sigma)\lambda}(\underline{0}) \\
& -\sum_{i=1}^{n}\pi_{i1}\left\{\delta^{(\gamma|(\alpha_1}\mathcal{K}^{\beta_1)|\delta)\cdots\alpha_n\beta_n}{}_{\mu\nu\rho\sigma}(\underline{0})\right\}.
\end{aligned}
$$

2-point flow equation: Weyl Scheme

Consider the effective action in Weyl scheme, this time using a fixed background:

$$S = 2\int_x \sqrt{g}\left(R_{\alpha\beta}c^{-1}R^{\alpha\beta} + sRc^{-1}R + \cdots\right),$$

where c is the cutoff function. The 2-point function is

$$\mathcal{S}^{\alpha\beta\gamma\delta}(p,-p) = c^{-1}\mathcal{S}_a^{\alpha\beta\gamma\delta}(p,-p) + (1+4s)\,c^{-1}\mathcal{S}_b^{\alpha\beta\gamma\delta}(p,-p),$$

which solves the 2-point flow equation provided that $\dot{(c^{-1})} = -p^4c^{-2}\dot{\Delta},$

and $s\dot{(c^{-1})} = -p^4c^{-2}\dot{\Delta}\left[4j(1+3s)^2 + (1+2s)(1+6s)\right].$

This is solved by $\Delta\left(p^2\right) = \dfrac{1}{p^4}c\left(p^2\right),$

and $j = -\dfrac{1}{4}\dfrac{1+4s}{1+3s}.$

2-point flow equation: Einstein scheme

Let us now consider an effective action that begin with the Einstein-Hilbert term:

$$S = 2\int_x \sqrt{g}\left(-R + \frac{1}{\Lambda^2} R_{\mu\nu} d\left(\frac{-\nabla^2}{\Lambda^2}\right) R^{\mu\nu} + \frac{j}{\Lambda^2} R d\left(\frac{-\nabla^2}{\Lambda^2}\right) R + \cdots\right).$$

We choose j = -1/2 so that the entire expression uses the same transverse projector, the flow equation then requires only that

$$\Lambda\partial_\Lambda\left(\frac{d}{\Lambda^2}\right) = -p^2\dot{\Delta}\left(\frac{d}{\Lambda^2} + \frac{1}{p^2}\right)^2.$$

Alternatively, j = -1/3 would have given us the same constraint, but then the 2- and (4+)-derivative terms would use different transverse projectors.

We can then write the "effective propagator" as

$$\Delta^{-1} = p^2 + d\frac{p^4}{\Lambda^2} = p^2 c^{-1}\left(\frac{p^2}{\Lambda^2}\right).$$

So we have the familiar form $\Delta = \dfrac{1}{p^2} c\left(\dfrac{p^2}{\Lambda^2}\right).$

Comments on additional regularization

Remembering that *SU(N|N)* regularization is required for gauge theories, it is natural to suggest a similar Pauli-Villars regularization for gravity.

The metric would be appended with wrong-statistics fermionic components, so the invariant interval is

$$ds^2 = dx^A g_{AB} dx^B,$$

where we would also use Grassmann coordinates:

$$x^A = (x^\mu, \theta^a).$$

This gives us D^2 fermionic degrees of freedom: $g_{\mu a} = -g_{a\mu}.$

These cancel bosonic degrees of freedom, of which there are *D*(*D*+1)/2 from the symmetric original metric, $g_{\mu\nu},$

and *D*(*D*-1)/2 from the new bosonic (antisymmetric) part, $g_{ab}.$

Cancellations between fermionic and bosonic modes in the UV would be used to provide the additional regularization. A spontaneous symmetry breaking would be required to decouple the extra degrees of freedom at the cutoff scale.

Summary

- The manifestly diffeomorphism invariant ERG is based on a generalization of the Polchinski flow equation for gravity.
- There is no need for gauge fixing or BRST ghosts. All results are independent of coordinates.
- It has both fixed-background and background-independent versions that give consistent results.
- We developed the formalism at tree-level and can exactly solve for *n*-point vertices iteratively, starting at the 2-point level.
- We have suggested how to proceed with the quantum construction.