Large curvature in single-metric approximations

(a conundrum and its solution)

Asymptotic safety seminar

14/11/16

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TRM, arXiv:1610.03081.

- Polynomial truncations are really only justified if $\bar{\mathfrak{R}} = \bar{R}/k^2 \ll 1$

- Go beyond with $f_k(\bar{\mathfrak{R}})$ approximations.

- Allows to explore e.g. singularities at finite scaled curvature, or behaviour when scaled curvature is even diverging...

It can happen that the chosen modified Laplacians:

$$\bar{\Delta} = -\bar{\nabla}^2 + \bar{E}$$

have a minimum eigenvalue which is positive.

$$\bar{\Delta}\, u = \lambda^2 u$$

$$\lambda \geq \lambda_{min} > 0$$

M. Demmel, F. Saueressig & O. Zanusso, 1401.5495, 1504.07656;
N. Ohta, R. Percacci & G. P. Vacca, 1511.09393;
K. Falls & N. Ohta, 1607.08460.

What does it mean to have $k < \lambda_{min}$?

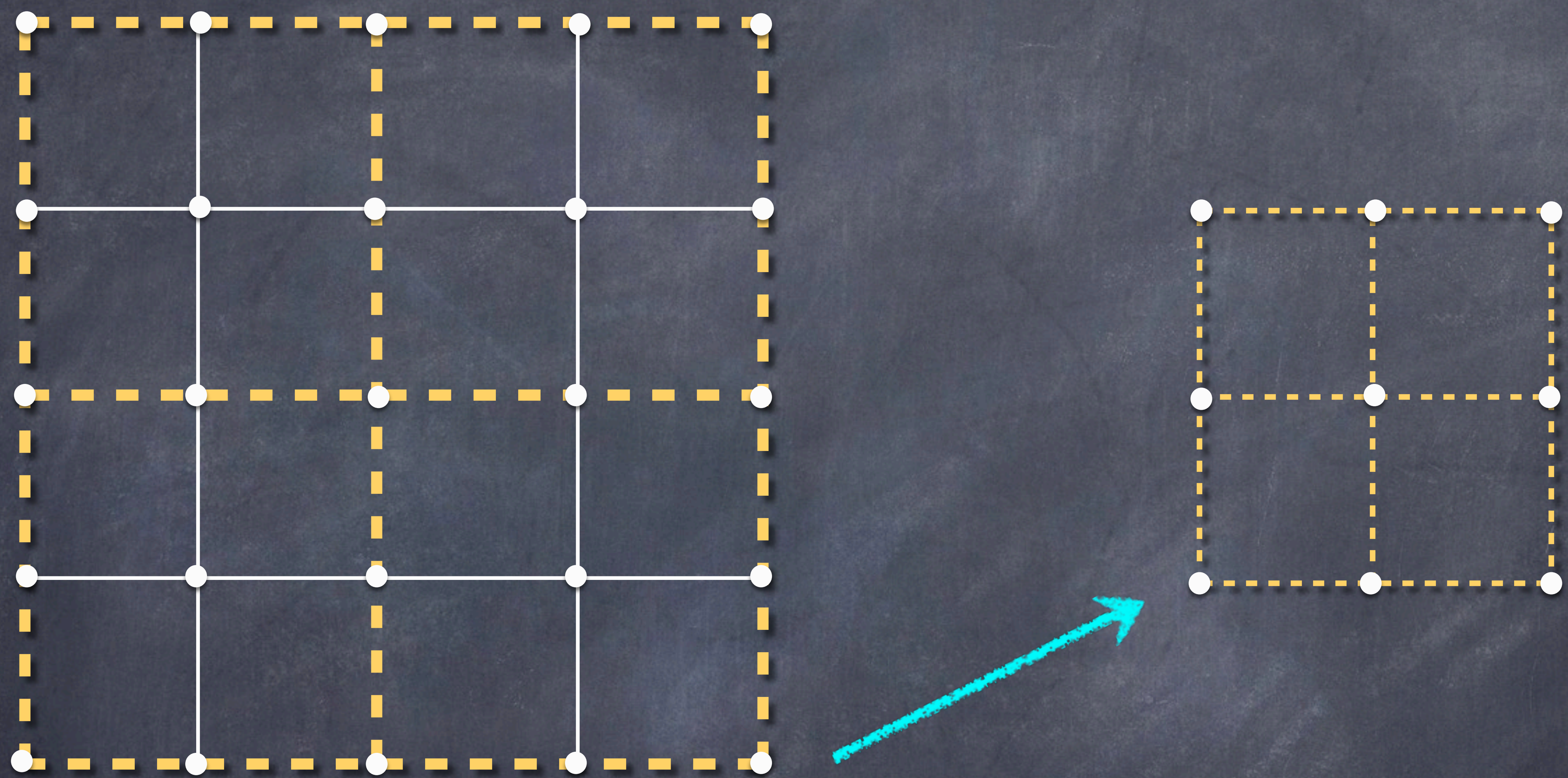
What does it mean to have $k < \lambda_{min}$?

- RG step must be well defined i.e. must be possible to lower k to any positive value, without encountering singularities.

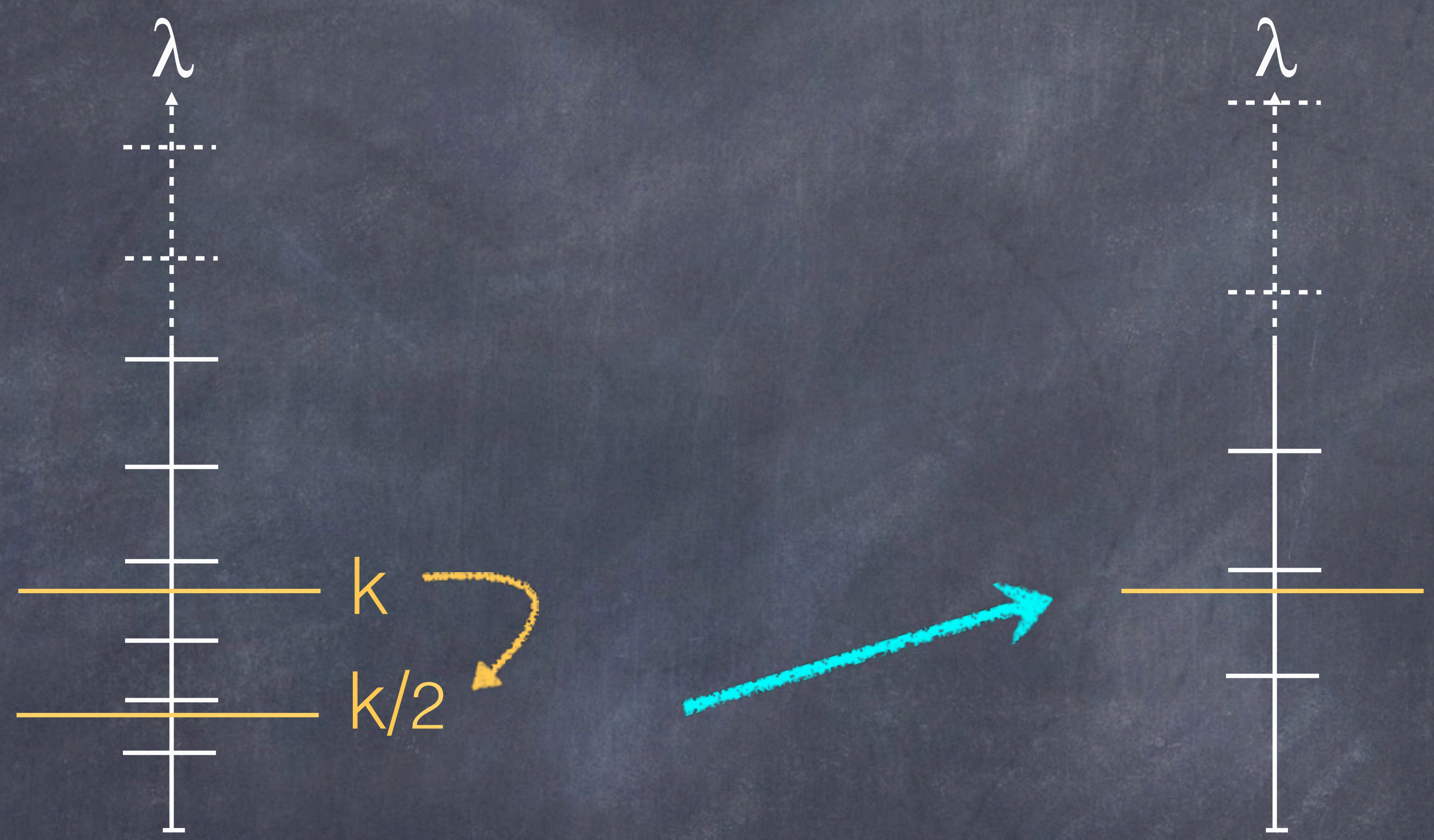
- Since $\bar{g}_{\mu\nu}$ fixed, this means $f_k(\bar{\mathfrak{R}})$ must be smooth no matter how large $\bar{\mathfrak{R}} = \bar{R}/k^2$ is taken.

- Full functional integral recovered only in the limit $k \to 0$

- But when $k < \lambda_{min}$ there is nothing left to cut off, so imposing smoothness conditions here is meaningless?



Wilsonian RG (e.g. fixed points) **not** possible on a finite lattice



Wilsonian RG (e.g. fixed points) **not** possible
when there is a lowest eigenvalue

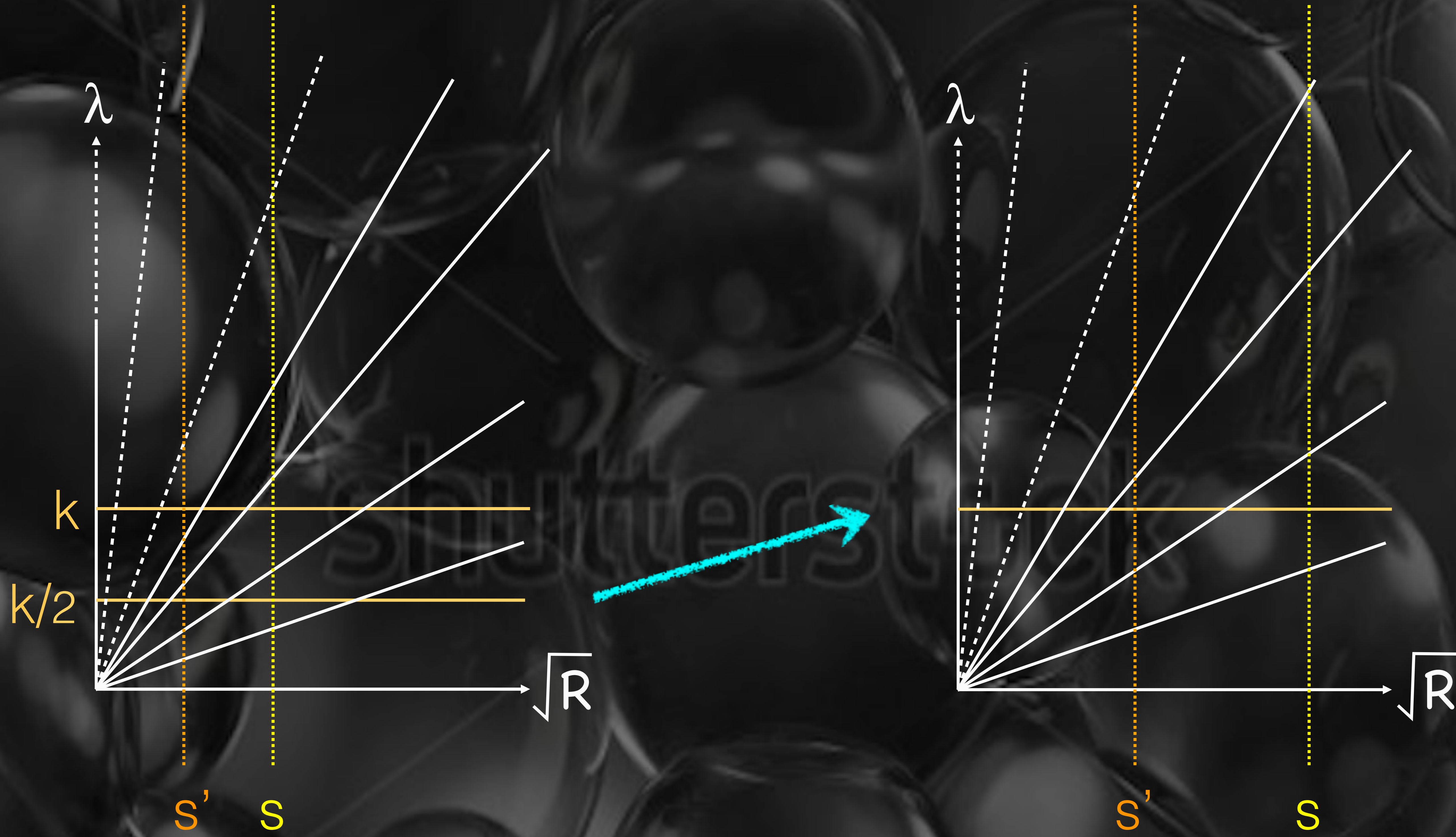
includes smooth cutoffs & the hyperboloid

<u>Background independence</u> is not respected.

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

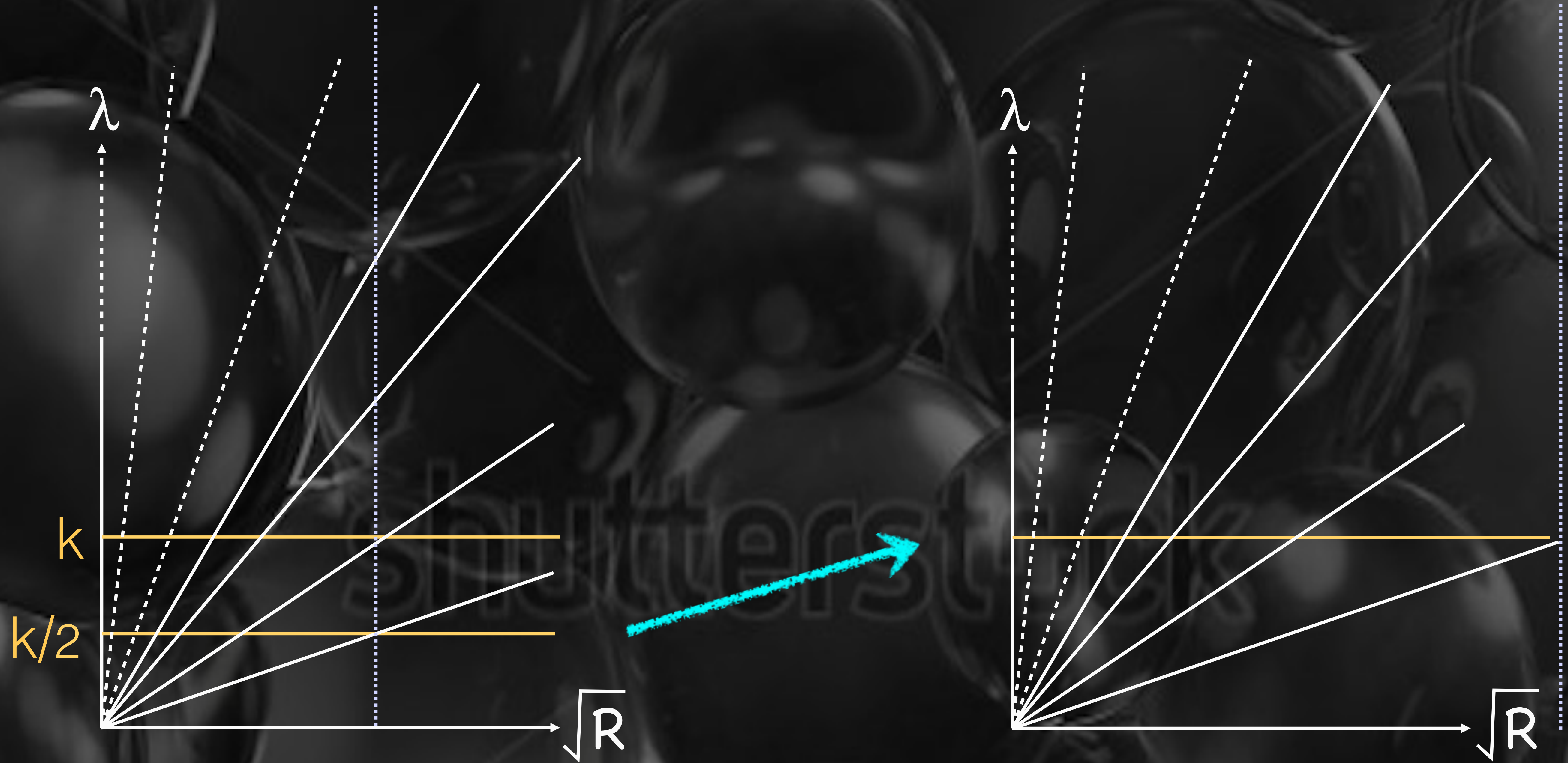
Results should depend only on this.

Even if we choose to restrict to spheres, we should still be integrating over their size



Wilsonian RG is restored with a continuous ensemble of spheres

provided that k is independent of $\bar{g}_{\mu\nu}$



Wilsonian RG is restored with a continuous ensemble of spheres
provided that k is independent of $\bar{g}_{\mu\nu}$

- this means $f_k(\bar{\mathfrak{R}})$ must be smooth no matter how large $\bar{\mathfrak{R}} = \bar{R}/k^2$ is taken.

Background independence

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

Results depend only on this.

However this is not really true in calculations!

... Wilsonian RG in QG?

We can recover by insisting that $h_{\mu\nu} \propto \bar{g}_{\mu\nu}$ does correspond to rescaling the sphere

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

$$\delta \bar{g}_{\mu\nu} = -2\varepsilon \bar{g}_{\mu\nu} \qquad \delta h_{\mu\nu} = +2\varepsilon \bar{g}_{\mu\nu}$$

$$\bar{\Gamma}^{\mu}_{\alpha\beta} = \frac{1}{2}\,\bar{g}^{\mu\nu}\left(\partial_\alpha \bar{g}_{\nu\beta} + \partial_\beta \bar{g}_{\nu\alpha} - \partial_\nu \bar{g}_{\alpha\beta}\right) \qquad [\bar{\nabla}_\mu, \bar{\nabla}_\nu]\xi^\beta = \bar{R}^{\alpha}{}_{\beta\mu\nu}\xi^\beta$$

$$\delta \bar{\Gamma}^{\mu}_{\alpha\beta} = 0 \qquad \delta \bar{\nabla}_\mu = 0 \qquad \delta \bar{R}^{\alpha}{}_{\beta\mu\nu} = 0$$

$$\boxed{\delta \bar{R} = +2\varepsilon \bar{R}}$$

Background rescaling invariance

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

$$\delta\bar{g}_{\mu\nu} = -2\varepsilon\bar{g}_{\mu\nu} \qquad \delta h_{\mu\nu} = +2\varepsilon\bar{g}_{\mu\nu}$$

$$\bar{\Gamma}^{\mu}_{\alpha\beta} = \frac{1}{2}\,\bar{g}^{\mu\nu}\left(\partial_\alpha\bar{g}_{\nu\beta} + \partial_\beta\bar{g}_{\nu\alpha} - \partial_\nu\bar{g}_{\alpha\beta}\right) \qquad [\bar{\nabla}_\mu, \bar{\nabla}_\nu]\xi^\beta = \bar{R}^{\alpha}{}_{\beta\mu\nu}\xi^\beta$$

$$\delta\bar{\Gamma}^{\mu}_{\alpha\beta} = 0 \qquad \delta\bar{\nabla}_\mu = 0 \qquad \delta\bar{R}^{\alpha}{}_{\beta\mu\nu} = 0$$

$$\boxed{\delta\bar{R} = +2\varepsilon\bar{R}}$$

$$h_{\mu\nu} = \underbrace{h^T_{\mu\nu} + \bar{\nabla}_\mu\xi_\nu + \bar{\nabla}_\nu\xi_\mu + \bar{\nabla}_\mu\bar{\nabla}_\nu\sigma}_{\text{invariant}} + \frac{1}{d}\bar{g}_{\mu\nu}h$$

$$h(x) = \bar{h} + h^{\perp}(x)$$

$$\delta h^{\perp} = +2\varepsilon h^{\perp} \qquad \delta\bar{h} = +2\varepsilon(\bar{h} + d)$$

$$\delta\bar{g}_{\mu\nu} = -2\bar{g}_{\mu\nu} \qquad \delta\bar{h} = +2(\bar{h}+d) \qquad \delta\left(1+\frac{\bar{h}}{d}\right) = 2\left(1+\frac{\bar{h}}{d}\right)$$

$$\hat{g}_{\mu\nu} = \left(1+\frac{\bar{h}}{d}\right)\bar{g}_{\mu\nu}$$

invariant

$$\sqrt{\hat{g}}\hat{R} = \sqrt{\bar{g}}\bar{R}\left(1+\frac{\bar{h}}{d}\right)^{d/2-1}$$

$$= \sqrt{\bar{g}}\bar{R} + \frac{d-2}{2d}\sqrt{\bar{g}}\bar{R}\bar{h} + \frac{(d-2)(d-4)}{8d^2}\sqrt{\bar{g}}\bar{R}\bar{h}^2$$

$$+ \frac{(d-2)(d-4)(d-6)}{48d^3}\sqrt{\bar{g}}\bar{R}\bar{h}^3 + O(\bar{h}^4)\,.$$

$$\delta \bar{g}_{\mu\nu} = -2\bar{g}_{\mu\nu} \qquad \delta h_{\mu\nu} = +2\bar{g}_{\mu\nu}$$

Gauge fixing

$$F_\mu = \bar{\nabla}_\rho h^\rho_\mu - \frac{1}{d}\bar{\nabla}_\mu h^\rho_\rho \qquad \delta F_\mu = 2F_\mu$$

$$h_{\mu\nu} = h^T_{\mu\nu} + \bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu + \bar{\nabla}_\mu \bar{\nabla}_\nu \sigma + \frac{1}{d}\bar{g}_{\mu\nu} h$$

picks out just gauge degrees of freedom: ξ_μ, σ

$$S_{GF} = \frac{1}{2\alpha}\int \sqrt{\bar{g}}\, \bar{g}^{\mu\nu} F_\mu F_\nu \qquad \boxed{\delta u = \frac{d - d_u}{2} u}$$

In Landau gauge limit $\alpha \to 0$, invariance if: $d_{\text{gauge}} = 6$

Similarly, ghost and auxiliary field bilinear actions **invariant** for suitable choice:

$$\boxed{\delta u = \frac{d - d_u}{2} u}$$

$$h_{\mu\nu} = h^T_{\mu\nu} + \bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu + \bar{\nabla}_\mu \bar{\nabla}_\nu \sigma + \frac{1}{d} \bar{g}_{\mu\nu} h$$

Not so for physical fields:

$$\delta h^T_{\mu\nu} = 0 \qquad \delta h = 2(h + d)$$

$$d_{h^T_{\mu\nu}} = d \qquad d_h = d - 4$$

(plus inhomogeneous)

For quantities that depend only on the background metric:

$\delta\bar{g}_{\mu\nu} = -2\bar{g}_{\mu\nu}$ equivalent to $\delta x^\mu = x^\mu$ plus $\delta Q = [Q]\,Q$

$$\Longrightarrow \delta T^{\beta_1\cdots\beta_p}_{\alpha_1\cdots\alpha_q} = (p-q)\,T^{\beta_1\cdots\beta_p}_{\alpha_1\cdots\alpha_q}$$

$$\delta\bar{R} = 2\bar{R} \qquad \delta\bar{\Delta} = 2\,\bar{\Delta}$$

$$P_k(\bar{\Delta}) = \bar{\Delta} + k^2 r(\bar{\Delta}/k^2) \qquad t = \ln(k/\mu)$$

$$\delta P_k(\bar{\Delta}) = 2\,P_k(\bar{\Delta}) - \partial_t P_k(\bar{\Delta})$$

$$\Gamma^{(2)}_u = \frac{1}{\sqrt{\bar{g}}(x)\sqrt{\bar{g}}(y)}\frac{\delta^2\Gamma}{\delta u(x)\,\delta u(y)} \qquad \bar{\mathcal{R}}_u = \bar{\Gamma}^{(2)}_u(P_k) - \bar{\Gamma}^{(2)}_u(\bar{\Delta})$$

$$\delta\bar{\Gamma}^{(2)}_u = (d_{\bar{\mathcal{R}}u} - \partial_t)\bar{\Gamma}^{(2)}_u \quad \text{and} \quad \delta\bar{\mathcal{R}}_u = (d_{\bar{\mathcal{R}}u} - \partial_t)\bar{\mathcal{R}}_u$$

$$d_{\bar{\mathcal{R}}u} = [\bar{\Gamma}^{(2)}_u] + p_u - q_u$$

$$h_{\mu\nu} = h^T_{\mu\nu} + \bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu + \bar{\nabla}_\mu \bar{\nabla}_\nu \sigma + \frac{1}{d} \bar{g}_{\mu\nu} h$$

$$\sim \frac{1}{2} \int \sqrt{\bar{g}}\, \bar{g}^{\alpha\mu} \bar{g}^{\beta\nu} h_{\alpha\beta} (-\bar{\nabla}^2) h_{\mu\nu}$$

$$d_{\bar{\mathcal{R}} h^T_{\mu\nu}} = 6$$

$$d_{\bar{\mathcal{R}}\mathrm{gauge}} = 6 \quad ✔$$

$$d_{\bar{\mathcal{R}} h} = 2$$

$$d_{h^T_{\mu\nu}} = d$$

$$d_h = d - 4$$

$$d_{\bar{\mathcal{R}} u} = [\bar{\Gamma}^{(2)}_u] + p_u - q_u$$

$$h_{\mu\nu} = h^T_{\mu\nu} + \bar\nabla_\mu \xi_\nu + \bar\nabla_\nu \xi_\mu + \bar\nabla_\mu \bar\nabla_\nu \sigma + \frac{1}{d}\bar g_{\mu\nu} h$$

$$\sim \frac{1}{2}\int \sqrt{\bar g}\, \bar g^{\alpha\mu} \bar g^{\beta\nu} h^T_{\alpha\beta}(-\bar\nabla^2) h^T_{\mu\nu}$$

$$d_{\bar{\mathcal{R}} h^T_{\mu\nu}} = 6$$

$$d_{\bar{\mathcal{R}}\mathrm{gauge}} = 6 \quad ✔$$

$$d_{h^T_{\mu\nu}} = d$$

$$S_{\bar{\mathcal{R}}} = \frac{1}{2}\int \sqrt{\bar g} \sum_u u \bar{\mathcal{R}}_u u$$

$$d_{\bar{\mathcal{R}} h} = 2$$

$$d_h = d - 4$$

$$\begin{aligned} \delta S_{\bar{\mathcal{R}}} &= 2d\int \sqrt{\bar g}\, \bar{\mathcal{R}}_h h + \frac{1}{2}\int \sqrt{\bar g} \sum_u u\, (d_{\bar{\mathcal{R}}u} - d_u - \partial_t) \bar{\mathcal{R}}_u\, u \\ &= 2d\int \sqrt{\bar g}\, \bar{\mathcal{R}}_h h - \frac{1}{2}\int \sqrt{\bar g} \sum_u u\, \dot{\bar{\mathcal{R}}}_u\, u + \frac{6-d}{2} \sum_{u = h^T_{\mu\nu}, h} \int \sqrt{\bar g}\, u \bar{\mathcal{R}}_u u\,. \end{aligned}$$

$$d_{\bar{\mathcal{R}}u} = [\bar\Gamma^{(2)}_u] + p_u - q_u$$

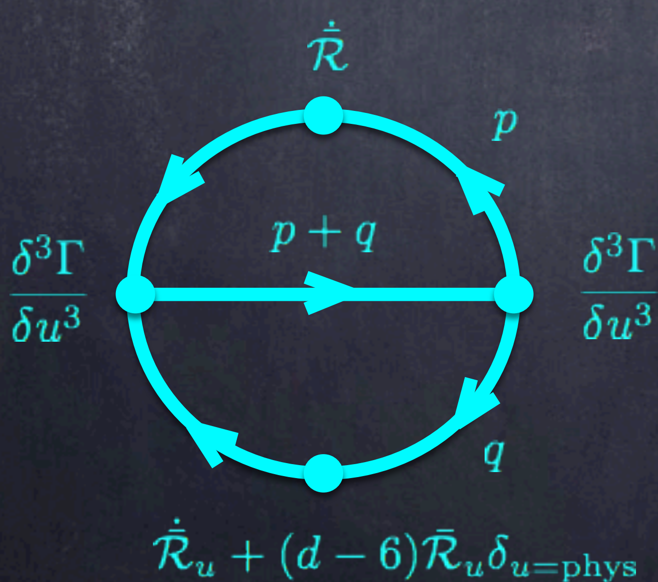
$$2\int \bar{g}_{\mu\nu}\,\frac{\delta\Gamma}{\delta\bar{g}_{\mu\nu}} - 2d\int\frac{\delta\Gamma}{\delta h} - \frac{1}{2}\sum_u (d-d_u)\int u\,\frac{\delta\Gamma}{\delta u}$$

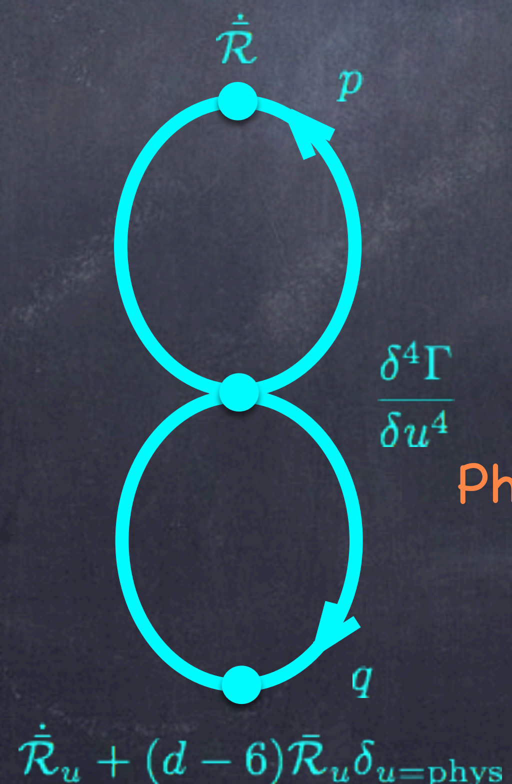
mWI

$$= \frac{1}{2}\sum_u \mathrm{tr}\Big[(\Gamma^{(2)}_u + \bar{\mathcal{R}}_u)^{-1}\,\dot{\bar{\mathcal{R}}}_u\Big] + \frac{d-6}{2}\sum_{u=h^T_{\mu\nu},h}\mathrm{tr}\Big[(\Gamma^{(2)}_u + \bar{\mathcal{R}}_u)^{-1}\,\bar{\mathcal{R}}_u\Big]$$

Flow eqn

$$\dot{\Gamma} = \frac{1}{2}\sum_u \mathrm{tr}\Big[(\Gamma^{(2)}_u + \bar{\mathcal{R}}_u)^{-1}\,\dot{\bar{\mathcal{R}}}_u\Big]$$





P. Labus, TRM, Z.H. Slade,
Phys. Rev. D**94** (2016), 024007; arXiv:1603.04772.

d=6

- Keep contribution from fluctuation fields to RHS but then discard.
- Make single metric approximation $\frac{\delta^2\Gamma}{\delta h_{\mu\nu}\delta h_{\alpha\beta}} \mapsto \frac{\delta^2\Gamma}{\delta\bar{g}_{\mu\nu}\delta\bar{g}_{\alpha\beta}}$
- Keep only $f_k(\bar{\mathfrak{R}})$. But retain $\bar{h}$

$$\boxed{2\int \bar{g}_{\mu\nu}\frac{\delta\Gamma}{\delta\bar{g}_{\mu\nu}} - 2d\frac{\partial\Gamma}{\partial\bar{h}} - 2\bar{h}\frac{\partial\Gamma}{\partial\bar{h}} =} \frac{1}{2}\sum_u \mathrm{tr}\left[(\bar{\Gamma}^{(2)}_u + \bar{\mathcal{R}}_u)^{-1}\dot{\bar{\mathcal{R}}}_u\right]$$

$$\boxed{\dot{\Gamma} =} \frac{1}{2}\sum_u \mathrm{tr}\left[(\bar{\Gamma}^{(2)}_u + \bar{\mathcal{R}}_u)^{-1}\dot{\bar{\mathcal{R}}}_u\right]$$

$$\delta\bar{g}_{\mu\nu} = -2\bar{g}_{\mu\nu} \qquad \delta\bar{h} = +2(\bar{h}+d) \qquad \delta k = k$$

- Keep contribution from fluctuation fields to RHS but then discard.

d=6

- Make single metric approximation $\frac{\delta^2\Gamma}{\delta h_{\mu\nu}\delta h_{\alpha\beta}} \mapsto \frac{\delta^2\Gamma}{\delta\bar{g}_{\mu\nu}\delta\bar{g}_{\alpha\beta}}$

- Keep only $f_k(\bar{\mathfrak{R}})$.

$$\Gamma = \hat{\Gamma}_{\hat{k}}[\hat{g}_{\mu\nu}]$$

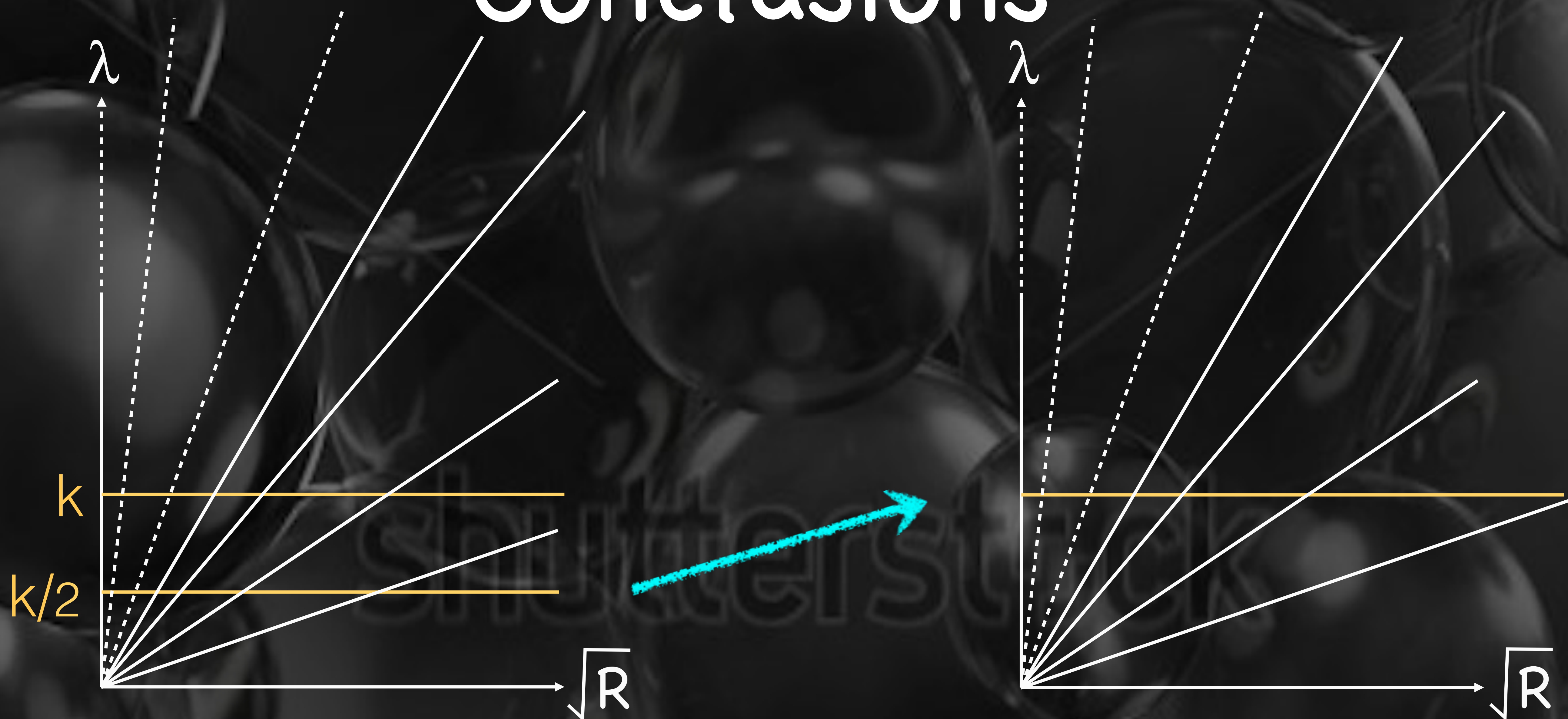
$$\hat{g}_{\mu\nu} = \left(1 + \frac{\bar{h}}{d}\right)\bar{g}_{\mu\nu} \qquad \hat{k} = k/\sqrt{1+\bar{h}/d}$$

$$\partial_{\hat{t}}\hat{\Gamma} = \frac{1}{2}\sum_u \mathrm{tr}\left[(\hat{\Gamma}^{(2)}_u + \hat{\mathcal{R}}_u)^{-1}\partial_{\hat{t}}\hat{\mathcal{R}}_u\right]$$

Exactly the same flow eqn, but with background scale invariant quantities.

$$\bar{g}_{\mu\nu} \mapsto \bar{g}_{\mu\nu}/a^2 \qquad \bar{h} \mapsto -d + (\bar{h}+d)a^2$$

Conclusions



- Treat backgrounds with different curvatures democratically (rigorous for $f_k(\bar{\mathfrak{R}})$ approx in d=6)
- Then Wilsonian RG (and k->0) has a meaning.
- $f_k(\bar{\mathfrak{R}})$ should be smooth for all $\bar{\mathfrak{R}} = \bar{R}/k^2$