Physical renormalisation schemes and asymptotic safety in quantum gravity

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Outline

- Motivation: understanding gauge and parameterisation dependencies in asymptotic safety e.g. near two dimensions.
- Aspects of the functional measure
- What's the source of unphysical dependencies?
- Physical renormalisation schemes.
- One-loop example.
- Application to quantum gravity near two dimensions.
- Conclusions.

Asymptotic Safety near two dimensions

- Simplest approximation to study asymptotic safety for gravity:

$$\beta_G = \varepsilon G - b\,G^2\,, \qquad \varepsilon = D-2\,. \qquad \text{Fixed point at } G_* = \varepsilon/b\,.$$

- But what is b?

- The result depends on the gauge… $\quad \nabla_\rho h^\rho{}_\mu - \frac{1+\beta}{D}\nabla_\mu h^\rho{}_\rho = 0$

$$b = \frac{2}{3}\frac{19-38\beta+13\beta^2}{(1-\beta)^2}\,, \qquad \text{for} \qquad g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}\,.$$

- … and the parameterisation (see Percacci and Vacca 2015):

$$b = \frac{2}{3}\frac{25-38\beta+19\beta^2}{(1-\beta)^2}\,, \qquad \text{for} \qquad g_{\mu\nu} = \bar{g}_{\mu\lambda}(e^h)^\lambda{}_\nu\,.$$

Asymptotic Safety near two dimensions

- Worse still a different result is obtained from the Gibbons-Hawking-York (GHY) boundary term:

$$S = -\frac{1}{16\pi G}\left(\int d^D x\sqrt{g}\, R + 2\int_\Sigma d^{D-1}y\sqrt{\gamma}\, K\right)$$

- From the GHY term one gets (Gatsman, Kallosh and Truffin ’78, Christensen and Duff ’78)

$$b = \frac{2}{3}$$

- but this result depends on the boundary conditions…

- Can we really give a physical meaning to the fixed point if it is not universal?

Renormalisation and scaling of observables

- Hypothesis: unphysical dependencies emerge when we consider unobservable local correlation functions.
- Idea: consider the renormalisation of observables directly to avoid unphysical results.
- Instead of correlation functions $\Lambda\partial_\Lambda \langle h_{\mu_1\nu_1}(x_1)...h_{\mu_n\nu_n}(x_n)\rangle$,
- we can look at the scaling of diffeomorphism invariant quantities e.g.

$$-\Lambda\partial_\Lambda \left\langle \int d^D x \sqrt{g} \right\rangle = d_{\mathcal{V}} \left\langle \int d^D x \sqrt{g} \right\rangle$$

- Λ denotes the UV cutoff scale.

Classically we have $-\Lambda\partial_\Lambda g_{\mu\nu} = -2 g_{\mu\nu}$ such that $d_{\mathcal{V}} = -D + \eta_{\mathcal{V}}$

The functional measure in quantum gravity

- The formal expression for the partition function:

$$\mathcal{Z} = \int d\mathcal{M}[\phi]\, e^{-\frac{1}{16\pi G}\int d^D x \sqrt{g}(2\bar{\lambda}-R)+\ldots}$$

- What is the field? $\phi^A = g_{\mu\nu}\,, \quad \phi^A = g^{\mu\nu}\,, \quad \phi^A = \sqrt{g}g^{\mu\nu}$ etc.

- Background field split: $g_{\mu\nu} = \bar{g}_{\mu\nu} + \phi_{\mu\nu}\,, \qquad g_{\mu\nu} = \bar{g}_{\mu\rho}(e^{\phi})^{\rho}{}_{\nu}$

- Physics should not depend on these choices.

- Measure should be invariant:

$$d\mathcal{M}[\phi] = \prod_a \frac{d\phi^a}{(2\pi)^{1/2}} V_{\text{diff}}^{-1}[\phi]\sqrt{|\det C_{ab}[\phi]|}\,, \qquad a = \{x, \mu\nu\}\,.$$

The functional measure in quantum gravity

- View fields as coordinates on the space of geometries
- Invariant line element: $\delta l^2 = C_{ab}\delta\phi^a\delta\phi^b$
- DeWitt metric:

$$C_{ab}\delta\phi^a\delta\phi^b = \frac{\mu^2}{32\pi G}\frac{1}{2}\int d^Dx\sqrt{g}(g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho} - g^{\mu\nu}g^{\rho\sigma})\delta g_{\mu\nu}\delta g_{\rho\sigma}$$

- Similarly the volume of diffeomorphisms comes with a volume element in terms of the metric:

$$\delta\ell^2_{\text{diff}} = \delta\xi^\alpha\delta\xi^\beta G_{\alpha\beta} = \frac{\mu_\epsilon^4}{16\pi G}\int d^Dx\sqrt{g}g_{\mu\nu}\epsilon^\mu\epsilon^\nu\,,$$

- Fradkin and Vilkovisky ‘73 have argued that the choice of measure should remove the strongest divergencies. This can be achieved by tuning the overall normalisation of the measure such that the divergencies,

$$\sim \Lambda^D\int d^Dx\sqrt{g}$$

  are absent. Later we will tune the ratio $\zeta_\epsilon \equiv \mu_\epsilon/\mu$ to achieve this.

- The two dimensional limit is singular for fixed Newtons constant. But a non-singular limit exists:

$$D\to 2\,,\quad G\propto D-2\,.$$

The source of parameterisation dependence

- Beta functions depend on how we choose to parameterise the physical degrees of freedom through the choice of field variables and the gauge fixing condition.

- These dependencies are due to the insertion of the source term in the functional integral which breaks both diffeomorphism and reparameterisation invariance

$$e^{W[J]} = \int \prod_n \frac{d\varphi^n}{(2\pi)^{1/2}} \sqrt{|\mathrm{sdet}\, \mathcal{C}_{nm}(\varphi)|}\, e^{-\mathcal{S}[\varphi] + J_n \varphi^n}$$

- In the effective action the source is given by the equations of motion

$$\frac{\delta}{\delta \varphi^n} \Gamma[\varphi] = J_n \qquad\qquad \varphi^n = \{g_{\mu\nu}(x), C_\mu(x), \bar{C}_\mu(x)\}$$

- As a consequence it is terms proportional to the equations of motion that give rise to the unphysical dependencies in beta functions.

The source of parameterisation dependence

- At one loop:

$$\Gamma[g_{\mu\nu}] = S[g_{\mu\nu}] + \frac{1}{2}\mathrm{STr}\log\left(\mathcal{C}^{-1}\cdot\mathcal{S}^{(2)}\right)$$

- This expression depends on both the action and the measure.

- If fact

$$\mathcal{S}^{(2)} = \mathcal{C}\cdot(-\nabla^2 + ...)$$

- Gauge and parameterisation dependence can be traced to terms in the Hessian which are proportional to the equations of motion.

The source of parameterisation dependence

- If we consider the one-loop terms which are divergent close to two dimensions the the RG flow of the bare action has the form:

$$\Lambda\partial_\Lambda S_\Lambda = \int d^Dx\sqrt{g}\left[B_0\Lambda^D + \Lambda^{D-2}\left(B_1 R + \bar{B}_1\left(R - \frac{2D}{D-2}\bar{\lambda}\right)\right)\right] + ...$$

- Where the coefficient of last term depends on the gauge fixing and choice of field variables.

- The one-loop beta functions of Newton's constant and he vacuum energy are:

$$\beta_G = (D-2)G + 16\pi(B_1 + \bar{B}_1)G^2\,, \qquad \beta_\lambda = -D\lambda + B_0 - \frac{2D}{D-2}\bar{B}_1 8\pi G\lambda\,.$$

- These assume that the metric has no anomalous dimension i.e. (after going to cutoff units):

$$-\Lambda\partial_\Lambda g_{\mu\nu} = -2g_{\mu\nu}$$

- Allowing for an anomalous dimension of the metric has the effect to modify the terms proportional to the equation of motion.

Field renormalisation conditions and preferred parameterisations

- Kawai and Ninomiya 89': We can choose a non-zero field renormalisation to remove the unphysical dependencies by enforcing a renormalisation condition e.g. we can specify that (within dimensional regularisation) the cosmological constant is not renormalised. This leads to

$$\beta_G = \varepsilon G - \frac{38}{3} G^2 \,, \qquad \beta_\lambda = -D\lambda \qquad\qquad \bar{B}_1 = 0$$

- However, this renormalisation condition is not unique. One can add matter fields and enforce that certain matter interactions are not renormalised by gravity instead.

- One can relate such renormalisation conditions to “preferred" (aka “physical”) gauges and/or field parameterisations e.g. by gauge fixing the conformal factor of the metric (see e.g. Percacci and Vacca 2015, Benedetti 2016) one gets the above beta function for Newton’s constant.

- Equally by choosing a field parameterisation where the volume element on spacetime is linear in the field one can remove the gauge dependence of the beta functions (KF 2015).

Field renormalisation conditions and preferred parameterisations

- To understand why these choices reproduce the same beta function note that to gauge fix the conformal factor we will not produce terms which involve the cosmological constant.

  - This is achieved using the exponential parameterisation and gauge fixing the trace of the fluctuation

$$\beta \to \infty\,, \quad \text{with } g_{\mu\nu} = \bar{g}_{\mu\lambda}(e^h)^\lambda{}_\nu\,. \qquad \nabla_\rho h^\rho{}_\mu - \frac{1+\beta}{D}\nabla_\mu h^\rho{}_\rho = 0$$

  - Such that variations of the spacetime volume vanish

$$\frac{\delta}{\delta h_{\mu\nu}}\sqrt{g} = 0$$

  - Equally we can pick a field variable such that

$$\frac{\delta^2}{\delta\phi_{\mu\nu}\delta\phi_{\rho\lambda}}\sqrt{g} = 0$$

  - These leads to: $b = \frac{2}{3}\frac{25 - 38\beta + 19\beta^2}{(1-\beta)^2}\Big|_{\beta\to\infty} = \frac{38}{3}$

Field renormalisation conditions and preferred parameterisations

- These choices have the effect that the scaling dimension of the volume is classical:

$$-\Lambda\partial_\Lambda \left\langle \int d^D x \sqrt{g} \right\rangle = -D \left\langle \int d^D x \sqrt{g} \right\rangle$$

- which can be read off the beta function:

$$\frac{\partial}{\partial \lambda}\beta_\lambda = -D$$

- However allowing for an anomalous dimension of the metric can modify this result i.e.

$$-\Lambda\partial_\Lambda g_{\mu\nu} = (-2+\eta_g)g_{\mu\nu}$$

- leads to an anomalous dimension of the volume

$$\frac{\partial}{\partial \lambda}\beta_\lambda = -D + \frac{1}{2}D\eta_g$$

Physical renormalisation schemes

- Build renormalisation schemes based on *reference observables* where instead of dependencies on unphysical parameters the beta functions depend explicitly on the anomalous dimension of the reference observables.

- Starting point is a regularised partition function in the absence of sources:

$$\mathcal{Z} = \int d\mathcal{M}_\Lambda[\phi] e^{-S_\Lambda[\phi]}$$

- This should be independent of the cutoff scale: $\partial_\Lambda \mathcal{Z} = 0$

- We then consider some reference observables to base the scheme on e.g.

$$\mathcal{O}(\mathcal{V}, V_1, V_2, ...)$$

- where here we have the spacetime volume and the volumes of the boundaries

$$\mathcal{V} = \int d^D x \sqrt{g}\,, \qquad V_n = \int_{\Sigma_n} d^{D-1} y \sqrt{\gamma}$$

Physical renormalisation schemes

- First we consider the case where we do not rescale or renormalise the fields and consider the condition:

$$-\Lambda\partial_\Lambda \langle \mathcal{O}(\mathcal{V}, V_1, V_2, ...)\rangle = 0$$

- in addition to the invariance of the partition function itself.

- This can be understood as a restriction on the RG flow of the action

$$S = \lambda(\Lambda)\mathcal{V} + \sum_m \rho_m(\Lambda) V_m + \sum_n \mathfrak{g}_n \mathcal{O}_n$$

- by imposing

$$\frac{\partial}{\partial\lambda}\Lambda\partial_\Lambda S = 0 = \frac{\partial}{\partial\rho_m}\Lambda\partial_\Lambda S$$

Physical renormalisation schemes

- If we then allow for an anomalous scaling of the metric

$$-\Lambda\partial_\Lambda g_{\mu\nu} = (-2+\eta_g) g_{\mu\nu}$$

- then have:

$$-\Lambda\partial_\Lambda \left\langle \int d^D x \sqrt{g} \right\rangle = d_{\mathcal{V}} \left\langle \int d^D x \sqrt{g} \right\rangle \,, \qquad d_{\mathcal{V}} \equiv -D + \eta_{\mathcal{V}} = -D + \frac{1}{2} D \eta_g \,.$$

$$-\Lambda\partial_\Lambda \left\langle \int d^d y \sqrt{\gamma} \right\rangle = d_V \left\langle \int d^{D-1} y \sqrt{\gamma} \right\rangle \,, \qquad d_V \equiv -D + 1 + \eta_V = -D + 1 + \frac{1}{2}(D-1)\eta_g \,.$$

- More generally we flow equation has a term which encodes the scaling of the field and is related to the scaling of reference observables:

$$\Lambda\partial_\Lambda S = d^a[\phi] \frac{\delta}{\delta\phi^a} S + \mathcal{F}\{S\} \,, \quad \text{with} \quad \frac{\partial}{\partial J}\mathcal{F}\{S\} = 0$$

$$-\Lambda\partial_\Lambda \phi^a = d^a[\phi] \,, \quad -\Lambda\partial_\Lambda \langle \mathcal{O} \rangle = \left\langle d^a[\phi] \frac{\delta}{\delta\phi^a} \mathcal{O} \right\rangle \,.$$

One-loop proper-time flow

- A one-loop proper-time flow equation can then be written down where the volumes are reference observables:

$$\Lambda\partial_\Lambda S = d^a \frac{\delta}{\delta\phi^a} S + \mathrm{Tr}_2[e^{-\Delta_2}] - 2\mathrm{Tr}_1[e^{-\Delta_1/\zeta_\epsilon^2}]\,.$$

- With the differential operators $\Delta_1 \epsilon_\mu = \left(-\nabla^2 - \frac{R}{D}\right)\epsilon_\mu\,, \quad \Delta_2 h_{\mu\nu} = -\nabla^2 h_{\mu\nu} - 2R_\mu{}^\rho{}_\nu{}^\sigma h_{\rho\sigma}\,.$

- This equation is independent of the way the physical degrees of freedom are parameterised by virtue of not breaking reparameterisation invariance.

- By using diffeomorphism invariant boundary conditions the flow equation preserves the flow of the bulk and boundary terms.

- We note the dependence on the parameter which controls the normalisation of the functional measure which can be tuned to ensure the strongest divergencies are absent.

One-loop beta functions

- The expansion of the heat kernels give

$$\mathrm{Tr}_2[e^{-\Delta_2}] - 2\mathrm{Tr}_1[e^{-\Delta_1}] = \frac{\frac{1}{2}D(D+1) - 2D\zeta_\epsilon^D}{(4\pi)^{\frac{D}{2}}} \int d^D x\sqrt{g}$$

$$+\frac{1}{(4\pi)^{\frac{D}{2}}} \int_\Sigma d^{D-1}y\sqrt{\gamma}\frac{\sqrt{\pi}}{2}\frac{1}{2}\left(D^2 - 4(D-2) - 3D + 4\zeta_\epsilon^{D-1}\right)$$

$$+\frac{1}{6}\frac{\frac{1}{2}D(D+1) - 6 - (2D+12)\zeta_\epsilon^{D-2}}{(4\pi)^{\frac{D}{2}}}\left(\int d^D x\sqrt{g}R + 2\int_\Sigma d^{D-1}y\sqrt{\gamma}K\right) + ...$$

- The one-loop beta functions are given by:

$$\beta_G = (D-2)\left(1 - \frac{\eta_\mathcal{V}}{D}\right)G - \frac{2}{3}\frac{\left(\frac{1}{2}D(D+1) - 6 - (2D+12)\zeta_\epsilon^{D-2}\right)}{(4\pi)^{\frac{D-2}{2}}}G^2\,,$$

$$\beta_\lambda = (-D + \eta_\mathcal{V})\,\lambda + \left(\frac{1}{2}D(D+1) - 2D\zeta_\epsilon^D\right)\frac{1}{(4\pi)^{\frac{D}{2}}}\,.$$

- Note we can put the constant term in the beta function for the vacuum energy to zero without effecting the beta function for Newton's constant in the two dimensional limit.

Quantum gravity near two dimensions

- In two dimensions the Einstein-Hilbert action enjoys Weyl invariance (i.e. local scale)

$$g_{\mu\nu} \to \Omega(x)^{-2} g_{\mu\nu}$$

- In consequence the hessian of the diffeomorphism invariant scalar, which is the sole diffeomorphism invariant degree of freedom is of the form

$$S^{(2)}_{ss} = -\frac{1}{D^2}\frac{(D-2)(D-1)}{32\pi G}\sqrt{g}(-\nabla^2 + \dots ,$$

- After canonically normalising this degree of freedom

$$s \to \sqrt{-\frac{32\pi G D^2}{(D-2)(D-1)}} s$$

- vertices have factors come with a factor $\sqrt{G/(D-2)}$ which means the loop expansion is in

$$G/\varepsilon \ll 1$$

- Since Weyl invariance plays a role here it is important to consider schemes that can preserve this symmetry. Including matter fields we can consider reference observables that are invariant under

$$g_{\mu\nu} \to \Omega(x)^{-2} g_{\mu\nu} \,, \qquad \psi \to \Omega(x)^{d_\psi} \psi$$

Matter interactions near two dimensions

- Here we will consider scalars and fermions with an interaction term which will play the role of the reference observable
$$\mathcal{O}[g_{\mu\nu}, \psi] = \int d^D x \sqrt{g} \mathcal{L}_{\rm int}(\psi)$$
- For different choices of the reference observable we have different schemes. The beta function for Newton's constant then takes a form that depends on the dimensionality of the reference observable
$$[\mathcal{O}] = d_0$$
- The anomalous dimension of the metric will then we related to that of the reference observable by
$$\eta_g(G) = -\frac{2}{d_0}\eta(G) \qquad -\Lambda\partial_\Lambda \langle \mathcal{O} \rangle = (d_0 + \eta) \langle \mathcal{O} \rangle$$
- At one-loop the beta functions are given by
$$\beta_G = \varepsilon \left(1 + \frac{\eta}{d_0}\right) G - \frac{2}{3}\left(25 + 3d_0 - c_\psi\right) G^2 \,, \qquad \beta_J = (d_0 + \eta) J \,.$$
- Here we have the central charge of the matter fields $\qquad c_\psi = N_S + N_F$
- Note that unless the classical dimension of the reference observable vanishes we break Weyl invariance. In the case that the symmetry is preserved the the beta function gives the result known from two dimensional quantum gravity. When the reference observable is the volume we recover our previous result.

Quantum gravity near two dimensions

- In each of our physical schemes the beta function near two dimensions still depends on the anomalous dimension of the metric. We can however relate different schemes and thus find that knowledge of one anomalous dimension in principle determines all other anomalous dimensions.

- Consider the expression for the one-loop beta function in terms of a gravitational central charge

$$\beta_G = \varepsilon G + \frac{2}{3}\left(c_g + c_\psi\right) G^2$$

- Then we write the anomalous dimension of the reference observable in terms of the unknown gravitational central charge

$$\eta = \frac{2}{3} d_0 \left(c_g + 3d_0 + 25\right) \frac{G}{\varepsilon}$$

- This does not yet give a conclusive picture and shows that generically the anomalous dimensions are large.

- In order to fix the unknown coefficient we now consider going to higher loop orders and understanding how the two dimensional limit can be taken.

Higher loops and Weyl invariance

- If we were to go to higher loops the expansion of the beta function for Newton's constant will be of the form

$$\beta_G = \varepsilon\left(1 - \frac{\eta_g}{2}\right)G - \frac{2}{3}\left(25 + 3d_0 - c_\psi\right)G^2 + G^2(b_2(d_0)\frac{G}{\varepsilon} + b_3(d_0)\frac{G^2}{\varepsilon^2} + ...)$$

- Now we consider the case where we preserve Weyl invariance in this case we can see that all higher loops will vanish.

- To see this we can make use of the fact that by looking only at observables we can freely pick how we parameterise the physical degrees of freedom without results being dependent on this choice. If we consider using the conformal gauge and using dimensionless matter fields:

$$g_{\mu\nu} = e^{2\sqrt{-8\pi G/\varepsilon}\sigma}\hat{g}_{\mu\nu} \qquad\qquad \psi = g^{-d_\psi/(2D)}\hat{\psi}$$

- The action then becomes quadratic in the conformal mode and hence the gravity contribution is one-loop exact in the two dimensional limit:

$$S[\phi] = -\frac{1}{16\pi G}\int d^Dx\sqrt{\hat{g}}\left[\hat{R}\left(1 + \varepsilon\sqrt{-8\pi G/\varepsilon}\sigma\right) - 8\pi G\hat{g}^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma\right]$$
$$+J\int d^Dx\sqrt{\hat{g}}\mathcal{L}(\hat{\psi}) + S_\psi[\hat{g}_{\mu\nu}, \hat{\psi}] + O(\varepsilon)$$

Two dimensional limit

- Thus for the Weyl invariant scheme we avoid the loop expansion and the beta function is given by

$$\beta_G = \varepsilon \left(1 - \frac{\eta_g}{2}\right) G - \frac{2}{3} \left(25 - c_\psi\right) G^2$$

- In this case the exact two dimensional beta function is given by

$$\beta_G = -\frac{2}{3} \left(25 - c_\psi\right) G^2$$

- We note that there are two fixed points at corresponding to ultraviolet and infrared limits

$$G \to \pm 0$$

- From point of view of dimensional regularisation, in two dimensions ε is an infrared regulator. Kawai, Kitazawa and Ninomiya '93 have argued that the two dimensional quantum gravity is reproduced at

$$G = -\frac{3}{2(25 - c_\psi)} \varepsilon \equiv G_{\mathrm{IR}}$$

- This we can interpret as an IR fixed point for $c_\psi < 25$

Two dimensional limit

- Kawai, Kitazawa and Ninomiya '93 showed that by starting with Einstein gravity in higher dimensions one can compute the scaling dimensions of observables in two dimensions to recover the KPZ exponents (Knizhnik, Polyakov, Zamolodchikov 88, David 88, Distler, Kawai 89)

$$d(G = G_{\mathrm{IR}}) \equiv -2\beta = -2(25 - c_\psi)\frac{1 - \sqrt{1 + 24\frac{1}{(25-c_\psi)}(\Delta_0 - 1)}}{12}$$

- Where in standard notation $\quad d_0(D = 2) \equiv -2(1 - \Delta_0)$

- If we expand in $1/(25 - c_\psi)$ we will then have the loop-expansion

$$d(G = G_{\mathrm{IR}}) = d_0 - 3\frac{d_0^2}{25 - c_\psi} + O(1/(25 - c_\psi)^2)$$

- Comparison with the one-loop anomalous dimension we obtained before

$$\eta = \frac{2}{3}d_0\left(c_g + 3d_0 + 25\right)\frac{G}{\varepsilon}$$

fixes the gravitational central charge

$$c_g = -25\,, \quad \beta_G = \varepsilon G - \frac{2}{3}(25 - c_\psi)G^2$$

Non-perturbative scaling exponents

- We can generalise the method of Kawai, Kitazawa and Ninomiya ’93 for general Newton’s constant.

- By choosing the conformal gauge in the parameterisation $$g_{\mu\nu} = \left(1 + \frac{\varepsilon}{2}\sqrt{-\frac{8\pi G}{\varepsilon}}\sigma\right)^{\frac{4}{\varepsilon}} \hat{g}_{\mu\nu}$$

- we canonically normalise the conformal factor such that around flat space-time we have the propagator
$$\mathfrak{G}(p^2) = \frac{1}{p^2}$$

- Each momentum integral is regulated in dimensional regularisation to give
$$\int \frac{d^D p}{(2\pi)^D} \mathfrak{G}(p^2) = -\frac{1}{2\pi}\frac{k^\varepsilon}{\varepsilon}$$

- This allows us to rewrite the partition function as a standard integral
$$\mathcal{Z}_\sigma(k) = \mathcal{N}\int_{-\infty}^{\infty} d(i\sigma) e^{\pi k^{-\varepsilon}\varepsilon\sigma^2}$$

- After rescaling the conformal mode by $\sigma \to -i\sigma/\varepsilon$, the expectation value of the reference observable is given by
$$\langle \mathcal{O}\rangle_k = \frac{1}{\sqrt{\varepsilon}\mathcal{Z}_\sigma(k)}\int d^D x \sqrt{\hat{g}}^{-\frac{d_0}{D}}\mathcal{L}(\hat{\psi})\int_{-\infty}^{\infty} d\sigma \exp\left\{\frac{1}{\varepsilon}\left(4(1-\Delta_0)\log\left(1+\sqrt{\frac{8\pi G}{\varepsilon}}\frac{1}{2}\sigma\right) - \pi k^{-\varepsilon}\sigma^2\right)\right\}$$

Non-perturbative scaling exponents

- We can then perform a saddle point approximation to compute the expectation value in the two dimensional limit

$$\langle \mathcal{O} \rangle_k \approx \int d^D x \sqrt{\hat{g}}^{-\frac{d_0}{D}} \mathcal{L}(\hat{\psi}) \exp\left\{ \frac{1}{\varepsilon} \left(4(1-\Delta_0) \log\left(1 + \sqrt{\frac{8\pi G}{\varepsilon}} \frac{1}{2}\sigma_0\right) - \pi k^{-\varepsilon}\sigma_0^2 \right) \right\}$$

- where $\sigma_0 = -\frac{1-\sqrt{1-16G\varepsilon^{-1}k^{\varepsilon}(1-\Delta_0)}}{2\sqrt{2\pi}\sqrt{G\varepsilon^{-1}}}$

- The anomalous dimension is then given by

$$k\partial_k \langle \mathcal{O} \rangle = \eta \langle \mathcal{O} \rangle$$

- Thus we obtain the scaling dimension

$$d(G) = d_0(D) + \frac{1 - \sqrt{1 - 16\frac{G}{D-2}(\Delta_0 - 1)}}{4\frac{G}{D-2}} + 2(1-\Delta_0)$$

Resumed beta function

- We now know the scaling dimensions of the reference observables and the exact beta function near two dimensions in the Weyl invariant scheme. Putting this information together one can find the resummed beta function in any of the physical schemes

$$\beta_G = \varepsilon\, G - \frac{2}{3}(25 - c_\psi)G^2 - \frac{\eta_g}{2}\varepsilon\, G - \frac{\varepsilon G}{2(\Delta_0 - 1)}\left(\frac{1 - \sqrt{1 - 16\frac{G}{\varepsilon}(\Delta_0 - 1)}}{4\frac{G}{\varepsilon}} + 2(1 - \Delta_0)\right)$$

$$\eta_g(G) = -\frac{2}{d_0}\eta(G)$$

- Expanding in Newton's constant then gives the loop-expansion

$$\beta_G = \varepsilon\, G - \frac{\eta_g}{2}\varepsilon\, G + \frac{2}{3}G^2\left(c_\psi - 6\Delta_0 - 19\right) - \frac{32\left(\Delta_0 - 1\right)^2 G^3}{\varepsilon} - \frac{320\left(\Delta_0 - 1\right)^3 G^4}{\varepsilon^2} + \ldots$$

- Where the anomalous dimension term cancels the higher loops leaving us with the simple beta function

$$\beta_G = \varepsilon\, G - \frac{2}{3}(25 - c_\psi)G^2 \qquad\qquad \beta_J = d(G)\, J \equiv (d_0 + \eta(G))J$$

Asymptotic Safety

- The non-perturbative beta functions obtained from the expansion near two dimensions have a UV fixed point

$$J_* = 0\,, \qquad G_* = \frac{3}{2}\frac{\varepsilon}{25 - c_\psi}$$

- At this fixed point we have the critical exponents for an interaction

$$\mathcal{L}_{\text{int}}(\psi) = \psi_S^{n_S}\left(\bar{\psi}_F \psi_F\right)^{\frac{n_F}{2}}$$

Given by

$$\theta \equiv -d(G_*) = -\frac{1}{6}(25 - c_\psi)\left(1 - \sqrt{1 - \frac{12\left(\frac{n_F}{2} - 2\right)}{25 - c_\psi}}\right) + \frac{1}{2}(D-2)(2 - n_F - n_S)$$

- From which we see that the real part is bounded from above such that there are only a finite number of relevant matter interactions.

Conclusions

- The dependence on how we parameterise the physical degrees of freedom can be removed from RG equations by concentrating on physical observables.

- This leads to beta functions which are then dependent on the anomalous dimension of reference observables.

- Applying this idea to quantum gravity near two dimensions we can complete the task of looking for the UV fixed point in this approximation. A non-perturbative expression for the scaling exponents of matter interactions is recovered in support of asymptotic safety.