The path integral of unimodular gravity

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Motivation

- JHEP **1606**, 115 (2016) (N. Ohta, R. Percacci, A. D. Pereira)

$$\int \mathcal{D}\gamma_{\mu\nu} e^{-S[g(\gamma)]}, \quad \int \mathcal{D}\gamma^{\mu\nu} e^{-S[g(\gamma)]}$$

$$\gamma_{\mu\nu} = g_{\mu\nu}(\sqrt{\det g_{\mu\nu}})^{-\frac{m}{1+dm}}, \quad \gamma^{\mu\nu} = g_{\mu\nu}(\sqrt{\det g_{\mu\nu}})^{\frac{m}{1+dm}}$$

$$g_{\mu\nu} = \gamma_{\mu\nu}(\det \gamma_{\mu\nu})^{m}, \quad g^{\mu\nu} = \gamma^{\mu\nu}(\det \gamma_{\mu\nu})^{-m}$$

- Not invertible for $m = -1/d$. Unimodular point: $\det g_{\mu\nu} = 1$.
- The divergences are invariant under a $\mathbb{Z}_2$ "duality" transformation between $\gamma_{\mu\nu}$ and $\gamma^{\mu\nu}$.

Motivation

- Unimodular Gravity (UG) : Cosmological constant as an integration constant.
- **QUG** $\stackrel{?}{=}$ **QGR** (Gravity only)
- Claim: **QUG** $=$ **QGR** with UG as a *SDiff* theory.
  (like $O(n)$ and $SO(n)$)

1 loop GR review

- $\hbar = 0$

$$\begin{aligned} G_{\mu\nu} + \Lambda g_{\mu\nu} &= 8\pi G_N^{(d)} T_{\mu\nu} \\ S_{Diff} &= \frac{1}{16\pi G_N^{(d)}} \int d^d x \sqrt{-g}\,(R - 2\Lambda) + S_m \\ T_{\mu\nu} &= -\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}} \end{aligned}$$

- $\hbar = 1$

$$\begin{aligned} Z_{Diff} &= \int \mathcal{D}h_{\mu\nu}\, e^{-S_{Diff}(\bar{g}_{\mu\nu} + h_{\mu\nu})} \\ \delta h_{\mu\nu} &= \bar{\nabla}_\mu \epsilon_\nu + \bar{\nabla}_\nu \epsilon_\mu \\ \bar{R}_{\mu\nu} &= \frac{1}{d} \bar{R} \bar{g}_{\mu\nu} \quad (\Lambda = 0) \end{aligned}$$

1 loop GR review

- Decomposition:

$$
\begin{aligned}
h_{\mu\nu} &= h^{\mathrm{TT}}_{\mu\nu} + \bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu + \left(\bar{\nabla}_\mu \bar{\nabla}_\nu - \frac{1}{d} \bar{g}_{\mu\nu} \bar{\nabla}^2 \right) \sigma + \frac{1}{d} \bar{g}_{\mu\nu} h \\
\epsilon_\mu &= \epsilon^{\mathrm{T}}_\mu + \bar{\nabla}_\mu \phi
\end{aligned}
$$

- Transformations:

$$
\delta h^{\mathrm{TT}}_{\mu\nu} = 0, \quad \delta \xi_\mu = \epsilon^{\mathrm{T}}_\mu, \quad \delta\sigma = 2\phi, \quad \delta h = 2\nabla^2 \phi
$$

- One obtains:

$$
Z_{Diff} = \left(\int \mathcal{D}\xi^{\mathrm{T}}_\mu \mathcal{D}\sigma \mathrm{Det}^{1/2} \Delta_{L_0} \right) e^{-S_0} \frac{\mathrm{Det}^{1/2}(\Delta_{L_1} - \frac{2\bar{R}}{d})}{\mathrm{Det}^{1/2}(\Delta_{L_2} - \frac{2\bar{R}}{d})}
$$

1 loop GR review

- The volume of *Diff*'s:

$$V(Diff) = \int \mathcal{D}\epsilon_\mu = \int \mathcal{D}\epsilon_\mu^{\mathrm{T}} \mathcal{D}\phi \mathrm{Det}^{1/2}\Delta_{L_0}$$

- Identification:

$$\xi_\mu \leftrightarrow \epsilon_\mu^{\mathrm{T}}, \quad \sigma \leftrightarrow \phi$$

- One obtains:

$$Z_{Diff} = V(Diff) e^{-S_0} \frac{\mathrm{Det}^{1/2}(\Delta_{L_1} - \frac{2\bar{R}}{d})}{\mathrm{Det}^{1/2}(\Delta_{L_2} - \frac{2\bar{R}}{d})}$$

SDiff's

- Definition:

$$SDiff : \nabla_\mu \epsilon^\mu = 0, \quad (\bar{\nabla}^2 \phi = 0)$$

- The volume of *SDiff*'s:

$$V(SDiff) = \int \mathcal{D}\epsilon_\mu \delta(\nabla_\mu \epsilon^\mu) = \int \mathcal{D}\epsilon_\mu^{\mathrm{T}} \mathrm{Det}^{-1/2} \Delta_{L_0}$$

- Identification:

$$h \leftrightarrow \bar{\nabla}^2 \phi \quad \Rightarrow \quad \boxed{V(Diff) = V(SDiff) \int \mathcal{D}h}$$

- $V(Q) = \int \mathcal{D}h$, $Q = Diff / SDiff$ = space of volume forms.

QUG as a *SDiff* theory

- By definition:

$$\delta_{SDiff} \det g = 2 \det g \nabla_\mu \epsilon^\mu = 0 \quad (g^{\mu\nu}\delta g_{\mu\nu} = 0)$$

- The volume form:

$$\mathrm{Vol}_d = \underbrace{\sqrt{-\det g}}_{\substack{\text{Covariant under } Diff'\text{s.} \\ \text{Invariant under } SDiff'\text{s}}} \; dx^0 \wedge dx^1 \wedge \ldots \wedge dx^{d-1}$$

QUG as a *SDiff* theory

- Exponential Parametrization:

$$g_{\mu\nu} = \bar{g}_{\mu\rho} \left(e^{h}\right)^{\rho}{}_{\nu}$$

- The volume form:

$$\mathrm{Vol}_d = \sqrt{-\det \bar{g}}\, e^{\mathrm{tr}\, h} dx^0 \wedge dx^1 \wedge \ldots \wedge dx^{d-1}$$

- We choose: $h_{\mu\nu} = h^{\mathrm{T}}_{\mu\nu}$.
- This is consistent with the identification $h \leftrightarrow \bar{\nabla}^2\phi$. Recall that for *SDiff*'s $\bar{\nabla}^2\phi = 0$.

QUG as a *SDiff* theory

- We define the action (ω is fixed):

$$S_{UG} = S_{SDiff} = \frac{1}{16\pi G_N^{(d)}} \int d^d x \, \omega R + S_m$$

- $\hbar = 1$

$$\begin{aligned} Z_{SDiff} &= \left(\int \mathcal{D}\xi_\mu^{\mathrm{T}} \right) e^{-S_0} \frac{\mathrm{Det}^{1/2}(\Delta_{L_1} - \frac{2\bar{R}}{d})}{\mathrm{Det}^{1/2}(\Delta_{L_2} - \frac{2\bar{R}}{d})} \frac{1}{\mathrm{Det}^{1/2}\Delta_{L_0}} \\ &= V(SDiff) e^{-S_0} \frac{\mathrm{Det}^{1/2}(\Delta_{L_1} - \frac{2\bar{R}}{d})}{\mathrm{Det}^{1/2}(\Delta_{L_2} - \frac{2\bar{R}}{d})} \end{aligned}$$

- $\boxed{Z_{Diff} = Z_{SDiff}}$

Hamiltonian analysis of UG

- $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$
- UG Lagrangian = Fierz-Pauli Lagrangian with $h = 0$

$$\mathcal{L}_{UG} = -\frac{1}{2}\partial_\alpha h_{\mu\nu}\partial^\alpha h^{\mu\nu} + \partial_\alpha h_\mu{}^\alpha \partial_\beta h^{\mu\beta} - \partial_\alpha h_\mu{}^\alpha \partial^\mu h + \frac{1}{2}\partial_\alpha h \partial^\alpha h$$

- Decomposition of $h_{\mu\nu}, \epsilon_\mu$

$$\begin{aligned}
h_{00} &= -2\phi \\
h_{0i} &= v_i = v_i^T + \partial_i v \\
h_{ij} &= h_{ij}^{\mathrm{TT}} + \partial_i \zeta_j + \partial_j \zeta_i + \left(\partial_i \partial_j - \frac{1}{d-1}\delta_{ij}\partial^2\right)\tau + \frac{1}{d-1}\delta_{ij} t \\
\epsilon_\mu &= (\epsilon_0, \epsilon_i^T + \partial_i \epsilon)
\end{aligned}$$

- $h = 0$ implies that $2\phi + t = 0$.

Hamiltonian analysis of UG

- Dirac-Bergmann algorithm

GR	UG
d Primary Constraints	$d-1$ Primary Constraints
d Secondary Constraints	$d-1$ Secondary Constraints
0 Tertiary Constraints	1 Tertiary Constraint
$2d$ First class constraints (FCC)	$2d-1$ First class constraints

- D.o.f in $d=4$

	GR	UG
(q,p)	2×10	2×9
FCC	8	7
D.o.f $=(q,p)$-2 FCC	2+2	2+2

- $\hbar = 1$

$$Z = \int \mathcal{D}q's\mathcal{D}p's \, \det M \, \delta(\phi's) \, \delta(\chi's) \, e^{-S[q's,p's]}$$

$$M_{ab} = \{\phi_a, \chi_b\}, \quad \phi's = \text{FCC}, \quad \chi's = \text{Gauge conditions}$$

- We have

$$\det M_{GR} = (\det \Box)^2, \quad \det M_{UG} = (\det \Box)^3$$

$$\phi_{GR} \neq \phi_{UG}$$

- Nevertheless: $\boxed{Z_{Diff} = Z_{SDiff} = (\det \Box_{TT})^{-1/2}}$.

Classical UG

$$\delta S_{UG} = \frac{1}{16\pi G_N^{(d)}} \int d^d x\, \omega (R_{\mu\nu} - 8\pi G^{(d)} \theta_{\mu\nu}) \delta g^{\mu\nu}, \quad (g_{\mu\nu}\delta g^{\mu\nu} = 0)$$

$$R_{\mu\nu} - 8\pi G^{(d)} \theta_{\mu\nu} = \psi g_{\mu\nu}, \quad \theta_{\mu\nu} = \frac{-2}{\omega} \frac{\delta S_m}{\delta g^{\mu\nu}}$$

$$\psi = \tilde{\psi} + 8\pi G^{(d)} f, \quad T_{\mu\nu} = \theta_{\mu\nu} + f g_{\mu\nu} \quad \text{such that} \quad \nabla_\mu T^{\mu\nu} = 0$$

$$R_{\mu\nu} - 8\pi G^{(d)} T_{\mu\nu} = \tilde{\psi} g_{\mu\nu} \quad \Rightarrow \boxed{R_{\mu\nu} - \frac{1}{d} g_{\mu\nu} R = 8\pi G^{(d)} (T_{\mu\nu} - \frac{1}{d} g_{\mu\nu} T)}$$

Classical UG

$$\boxed{R_{\mu\nu} - \frac{1}{d}g_{\mu\nu}R = 8\pi G^{(d)}(T_{\mu\nu} - \frac{1}{d}g_{\mu\nu}T)}$$

$$\nabla^{\mu}R_{\mu\nu} - \frac{1}{d}g_{\mu\nu}\nabla^{\mu}R = -\frac{8\pi G^{(d)}}{d}g_{\mu\nu}\nabla^{\mu}T, \quad \nabla^{\mu}G_{\mu\nu} = 0$$

$$\nabla_{\mu}\left(\frac{d-2}{2d}R + 8\pi G^{(d)}T\right) = 0$$

Then

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N^{(d)}T_{\mu\nu}$$

Λ as an integration constant.

UG as a Weyl invariant theory

$$S = \frac{1}{16\pi G_N^{(d)}} \int d^d x \sqrt{-g} R$$

$$g_{\mu\nu} = \gamma_{\mu\nu} \left(\frac{|\gamma|}{\omega^2} \right)^{-1/d}$$

$$S_{UG} = Z_N \int d^d x \, |\gamma|^{1/d} \omega^{\frac{d-2}{d}} \left[R[\gamma] + \frac{(d-1)(d-2)}{4d^2} \left(|\gamma|^{-1} \nabla |\gamma| - 2\omega^{-1} \nabla \omega \right)^2 \right]$$

UG and the Cosmological Constant Problem

$$\text{GR} = \begin{cases} G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N^{(d)} T_{\mu\nu} \\ S_{Diff} = \frac{1}{16\pi G_N^{(d)}} \int d^d x \sqrt{-g}\,(R - 2\Lambda) + S_m \end{cases}$$

$$\text{UG} = \begin{cases} G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N^{(d)} T_{\mu\nu} \\ S_{Diff} = \frac{1}{16\pi G_N^{(d)}} \int d^d x\, \omega R + S_m \end{cases}$$

$$S_{Diff} \supset -\int d^d x \sqrt{-g}(2\Lambda + V(\phi))$$

$$S_{SDiff} \supset -\int d^d x\, \omega V(\phi)$$

Thank you