BOUNCING AND EMERGENT UNIVERSES FROM HAMILTONIAN ANALYSIS OF ASYMPTOTICALLY SAFE QUANTUM GRAVITY

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Outline of the talk

- Introduction: Quantum Gravity.
- Short review of Dirac's constraint analysis
- RG improved modified Einstein-Hilbert Action (Reuter-Weyer)
- Hamiltonian Analysis of Asymptotically Safe Quantum Gravity
- Hamiltonian Analysis of Brans-Dicke Theory
- Cosmology at Sub-Planckian Era: Bouncing and Emergent Universes.
- Conclusions.

QUANTUM GRAVITY

- Einstein General Relativity works quite well for distances $l \gg l_{Pl}$ (=Planck lenght).

- Singularity problem and the quantum mechanical behavour of matter-energy at small distance suggest a quantum mechanical behavour of the gravitational field (Quantum Gravity) at small distances (High Energy).

- Many different approaches to Quantum Gravity: String Theory, Loop Quantum Gravity, Non-commutative Geometry, CDT, Asymptotic Safety etc.

- General Relativity is considered an effective theory. It is not pertubatively renormalizable (the Newton constant G has a $(\text{lenght})^{-2}$ dimension)

REUTER-WEYER ACTION PROPOSAL

(Phys. Rev. D 69 2004)

- The modified Einstein Hilbert action has external, non-geometrical fields G(x) and $\Lambda(x)$, determined by RG and assumed independent by the metric tensor g.

$$S_{mEH}[g, G(x), \Lambda(x)] \equiv \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(\frac{R}{G(x)} - 2\frac{\Lambda(x)}{G(x)}\right)$$

- Notice that if one sees $\phi(x)$ as the inverse of the Newton constant $\phi(x) \equiv \frac{1}{G(x)}$ the previous action is Brans-Dicke theory of gravity with an external Brans-Dicke field

- Reuter-Weyer stress that one should find extra integrability condition on the modified Einstein equations, that puts constraints on $G(x)$ and $\Lambda(x)$, or further constraints on the cut-off $k(x)$ identification (besides the symmetry of the systems etc).
- In Homogeneous and Isotropic symmetrical cases $G = G(t)$ and $\Lambda = \Lambda(t)$

ADM FORMALISM WITH G AND Λ VARIABLE

- Bonanno et al (CGQ 21 2004) proposed an Hamiltonian Analysis of the previous action using initially a York boundary term. Later the considered G and Λ as dynamical variables, modified the boundary term ending in a dynamic with first class constraints.

- On the ADM decomposition M=$\mathcal{R}\times\Sigma$ the covariant metric tensor is

$$g=-(N^2-N_iN^i)dt\otimes dt+N_i(dx^i\otimes dt+dt\otimes dx^i)+h_{ij}dx^i\otimes dx^j$$

- The Einstein-Hilbert action with York boundary term is

$$S_{ADM}[h_{ij},N,N^i]=\frac{1}{16\pi}\int_{R\times\Sigma}dtd^3x\sqrt{h}N\frac{1}{G(t,x)}\left({}^4R-2\Lambda(t,x)\right)+\frac{1}{8\pi}\int_{\partial M}\frac{K\sqrt{h}}{G(t,x)}d^3x$$

the York term, as it is well, know is added in order to have a “differentiable action”.

DIRAC′ S CONSTRAINT THEORY

- In general if one has a Lagrangian L, the conjugate momenta are defined

$p_i = \frac{\partial L}{\partial \dot{q}^i}$; for "constrained systems" (gauge theories for example) $det \left| \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \right| = 0$

therefore the Legendere map defines a relation between the p and q coordinates, which is called primary constraints $\phi_i(q,p) = 0$ (this relation defines the "constraint surface"Σ). One uses the Dirac notation $\phi_i(q,p) \approx 0$, to say that these functions are defined on all the space (q,p).

- The canonical Hamiltonian H_c is defined on the primary constraint surface

$$H_c(q,p) = p_i \dot{q}^i - L\big|_\Sigma$$

- The "effective Hamiltonian" $\tilde{H}$ is defined on all space (q,p) as

$$\tilde{H} = H_c + \lambda^i \phi_i(q,p)$$

DIRAC′ S CONSTRAINT THEORY

- One needs to impose that the dynamic stays on the constraint surface

$$\dot{\phi}^i = \{\phi^i, \tilde{H}\} \approx 0$$

- In case previous equation is not identically zero, there exists secondary constraints $\chi^j_{II}(q,p)$

$$\chi^j_{II} = \{\phi^j, \tilde{H}\} \approx 0$$

- Therefore one has the total Hamiltonian H_T that is a linear combination of the primary and secondary constraints

$$H_T = H_c + \lambda_i \phi^i(q,p) + \tilde{\lambda}_j(q,p)\chi^j_{II}(q,p)$$

- and so on…The procedure stops when, at certain stage, the constraints are identically preserved. Beside primary, secondary etc. the constraints are classified in first class constraints and second class. First class contraints have zero Poisson brackets with all other constraints. The remaining constraints are second class. First class constraints are generators of either gauge symmetries or diffeomorphisms

CONSTRAINT ANALYSIS OF EINSTEIN GENERAL RELATIVITY

- Einstein GR-action in ADM variable and York boundary term is

$$S_{ADM}[h_{ij}, N, N^i] = \frac{1}{16\pi G}\int_{M=R\times\Sigma} dt d^3x \sqrt{h} N \left({}^4R - 2\Lambda\right) + \frac{1}{8\pi G}\int_{\partial M} K d^3x$$

- The primary constraints are

$$\pi = \frac{\partial \mathcal{L}_{ADM}}{\partial \dot{N}} \approx 0 \quad \pi_i = \frac{\partial \mathcal{L}_{ADM}}{\partial \dot{N}^i} \approx 0$$

- The canonical Hamiltonian density H_c is

$$\mathcal{H}_c = N\mathcal{H} + \mathcal{H}_i N^i$$

where the Hamiltonian constraint $\mathcal{H}$ and the momentum constraints $\mathcal{H}_i$ are

$$\mathcal{H} = (16\pi G) G_{ajkl}\pi^{aj}\pi^{kl} - \frac{\sqrt{h}}{16\pi G}\left({}^3R - 2\Lambda\right) \qquad \mathcal{H}_i = -2\nabla_j \pi_i^j$$

$$G_{abcb} = \frac{1}{2\sqrt{h}}\left(h_{ac}h_{bd} + h_{ad}h_{bc} - h_{ab}h_{cd}\right)$$ is the DeWitt supermetric

CONSTRAINT ANALYSIS OF EINSTEIN GENERAL RELATIVITY

- The effective Hamiltonian density $\tilde{\mathcal{H}}$ is

$$\tilde{\mathcal{H}} = \lambda\pi + \lambda_i\pi^i + N\mathcal{H} + N_i\mathcal{H}^i$$

- Preserving the primary constraints, one gets the secondary constraints

$$\dot{\pi} = \{\pi, \int d^3x\tilde{\mathcal{H}}\} = -\mathcal{H} = 0; \quad \dot{\pi}_i = \{\pi_i, \int d^3x\tilde{\mathcal{H}}\} = -\mathcal{H}_i = 0$$

- The previous Hamiltonian constraint and momentum constraints are first class and, then, generators of time diffeomorphism and space diffeomorphism on the three-surfaces Σ. In fact for the ”space” part

$$\{\pi^{ij}(x), \int d^3yN^l\mathcal{H}_l\} = \mathcal{L}_{\mathbf{N}}\pi^{ij}(x) \qquad \{h_{ij}(x), \int d^3yN^l\mathcal{H}_l\} = \mathcal{L}_{\mathbf{N}}h_{ij}(x)$$

ADM FORMALISM WITH G AND Λ VARIABLE

- Assuming that Σ has no boundary, that is $\partial\Sigma=0$, the ADM action S_{ADM} becomes

$$S_{ADM}[h_{ij},N,N^i]=\frac{1}{16\pi}\int_{R\times\Sigma}\underbrace{\left[\frac{N\sqrt{h}}{G}(K_{ij}K^{ij}-K^2+{}^{(3)}R-2\Lambda)-2\frac{G_{,0}}{G^2}K\sqrt{h}+2\frac{G_{,i}f^i}{G^2}\right]}_{\mathcal{L}_{ADM}}dtd^3x$$

- The momentum densities π_{ij} results quite complicated

$$\pi^{ij}=\frac{\partial\mathcal{L}_{ADM}}{\partial\dot{h}_{ij}}=-\frac{\sqrt{h}}{16\pi G}\left(K^{ij}-h^{ij}K\right)+\frac{\sqrt{h}\,h^{ij}}{16\pi NG^2}\left(G_{,0}-G_{,k}N^k\right)$$

- Their form suggests that one can define a new momentum variable $\tilde{\pi}_{ij}$

$$\tilde{\pi}^{ij}=\pi^{ij}-\frac{\sqrt{h}\,h^{ij}}{16\pi NG^2}\left(G_{,0}-G_{,k}N^k\right)=-\frac{\sqrt{h}}{16\pi G}\left(K^{ij}-h^{ij}K\right)$$

- One can prove that the following transformation of variables is canonical

$$(N,N^i,h_{ij},\pi,\pi_i,\pi^{ij})\longmapsto(N,N^i,h_{ij},\pi,\pi_i,\tilde{\pi}^{ij})$$

HAMILTONIAN FORMALISM

- This is a system with Dirac's constraints, in fact the primary constraints are

$$\pi = \frac{\delta \mathcal{L}_{ADM}}{\delta \dot{N}} \approx 0 \qquad \pi_i = \frac{\delta \mathcal{L}_{ADM}}{\delta \dot{N}^i} \approx 0$$

- The total Hamiltonian H_T is then

$$H_T = \int_\Sigma \left(\lambda \pi + \lambda^i \pi_i + \mathcal{H}_{ADM} \right) d^3x$$

λ and λ_i being, following Dirac's constraint theory, Lagrange multipliers.

- The Hamiltonian density is defined as

$$\mathcal{H}_{ADM} \equiv \pi^{ab} \dot{h}_{ab} - \mathcal{L}_{ADM}$$

ADM FORMALISM WITH G AND Λ VARIABLE

- Preserving primary constaints $\pi \approx 0 \quad \pi_i \approx 0$, one gets the secondary constaints: the Hamiltonian constraint $\mathcal{H}$ and the and the momenta constraints $\mathcal{H}_i$ (*H_T* is the total Hamiltonian)

$$\dot{\pi} = \{\pi, H_T\} = \mathcal{H} \approx 0 \qquad \dot{\pi}_i = \{\pi_i, H_T\} = \mathcal{H}_i \approx 0$$

- Therefore one gets

$$\mathcal{H} = (16\pi G) G_{abcd}\tilde{\pi}^{ab}\tilde{\pi}^{cd} - \frac{\sqrt{h}({}^3R - 2\Lambda)}{16\pi G} - \frac{\sqrt{h}(G_{,0} - G_{,k}N^k)\bar{\nabla}_a N^a}{8\pi G^2 N^2} - \nabla_i \left(\frac{G_{,i}\sqrt{h}h^{ij}}{8\pi G^2} \right)$$

- And the momenta constraints

$$\mathcal{H}_i = -2\bar{\nabla}^a\tilde{\pi}_{ai} + \frac{\sqrt{h}(-G_{,i})\bar{\nabla}_a N^a}{8\pi G^2 N} - \sqrt{h}\bar{\nabla}_i \left(\frac{G_{,0} - G_{,k}N^k}{8\pi G^2 N} \right)$$

HAMILTONIAN FORMALISM

- The expressions of the previous constraints look quite complicated. One can start checking if the momentum constraints $\mathcal{H}_i$ are still the generators of Space diffeomorphisms

$$\{h_{ij}, \int d^3x \tilde{N}^i \mathcal{H}_i\} = \mathcal{L}_{\tilde{\mathbf{N}}} h_{ij}$$

- Repating the same calculation for the momenta

$$\{\tilde{\pi}_{ij}, \int d^3x \tilde{N}^i \mathcal{H}_i\} = \int d^3x \mathcal{L}_{\tilde{\mathbf{N}}} \tilde{\pi}_{ij} + \bar{\nabla}_a \left[\frac{\tilde{N}^s}{2} \left(\frac{G_{,s}}{8\pi G^2 N} \right) N^a h^{ij} \sqrt{h} \right]$$

The momentum constraints are still the generators of the diffeomorphism transformation on the three spatial surfaces if G(x) is not dependent by the spatial coordinates. Again G=G(t).

HAMILTONIAN FORMALISM

- The previous results means that the ADM starting metric

$$g = -(N^2 - N_i N^i) dt \otimes dt + N_i (dx^i \otimes dt + dt \otimes dx^i) + h_{ij} dx^i \otimes dx^j$$

is not working. One needs to start from the following ADM metric in *Gausiian normal coordinates*

$$g = -N^2(t) dt \otimes dt + h_{ij} dx^i \otimes dx^j$$

- The ADM Hamiltonian density $\mathcal{H}_{ADM}$ reduces to

$$\mathcal{H}_{ADM} = N \left((16\pi G) G_{abcd} \tilde{\pi}^{ab} \tilde{\pi}^{cd} - \frac{\sqrt{h}({}^{(3)}R - 2\Lambda)}{16\pi G} \right)$$

HAMILTONIAN FORMALISM

- Therefore the associated Hamiltonian constaints $\mathcal{H}$ is

$$\mathcal{H} = \left((16\pi G) G_{abcd} \tilde{\pi}^{ab} \tilde{\pi}^{cd} - \frac{\sqrt{h} ({}^{(3)}R - 2\Lambda)}{16\pi G} \right)$$

- There are not momentum constraints $\mathcal{H}_i$

- The constraints algebra is closed without any requirement on G(x). This means that G(x) can be a generic function of the space-time coordinates.

BRANS-DICKE THEORY

- Previous Hamiltonian analysis triggers the question why the constraint algebra looks so complicated and do not close.

- To answer this question one can consider a Brans-Dicke theory in which $\phi(x)$ is a dynamical variable and a York boundary term

$$S = \frac{1}{4q^2}\left[\int_M d^4x\sqrt{-g}\left(\phi^2\ {}^{(4)}R + 4g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - U(\phi)\right) + 2\int_{\partial M} d^3x\sqrt{h}\phi^2 K\right]$$

- The ADM decomposition of the previous action is

$$S_{ADM} = \frac{1}{4q^2}\int_{\Sigma\times \mathrm{t}} d^3x dt N\sqrt{h}\Big(\phi^2\ {}^{(3)}R + \phi^2 K_{ij}K^{ij} - \phi^2 K^2 - \frac{4}{N^2}(\partial_0\phi)^2 +$$
$$\frac{8}{N^2}N^i\partial_0\phi\partial_i\phi + 4\partial_i\phi\partial^i\phi - 4\frac{N^iN^j}{N^2}\partial_i\phi\partial_j\phi + \frac{4}{N}\phi\phi_{,0}K - \frac{4}{N\sqrt{h}}\phi\phi_{,i}f^i - U(\phi)\Big)$$

$$f^i = \sqrt{h}\left(KN^i - h^{ij}N_{,j}\right)$$

ADM ANALYSIS OF BRANS-DICKE THEORY

- The definition of the momenta gives

$$\pi = \frac{\partial \mathcal{L}_{ADM}}{\partial \dot{N}} \approx 0 \qquad \pi_i = \frac{\partial \mathcal{L}_{ADM}}{\partial \dot{N}^i} \approx 0$$

$$\pi_\phi = \frac{\partial \mathcal{L}_{ADM}}{\partial \dot{\phi}} = -\frac{2\sqrt{h}}{4q^2 N}\left(\partial_0\phi - N^i\partial_i\phi - \frac{N}{2}\phi K\right) \qquad \pi^{ij} = \frac{\partial \mathcal{L}_{ADM}}{\partial \dot{h}_{ij}} = -\frac{\sqrt{h}}{4q^2}\phi^2 K^{ij} + \frac{1}{4}\phi\pi_\phi h^{ij}$$

- The first two momenta are primary constraints, an the total Hamiltonian H_T is

$$H_T = \int d^3x \left(\lambda\pi + \lambda^i\pi_i + N\mathcal{H} + N^i\mathcal{H}_i\right)$$

being λ and λ^i are Lagrange multipliers.

ADM ANALYSIS OF BRANS-DICKE THEORY

- As in Einstein General Relativity one has a momentum constraints $\mathcal{H}_i$ and Hamiltonian constraint $\mathcal{H}$

$$\mathcal{H}_i = -2\nabla_j \pi_i^j + \pi_\phi \partial_i \phi$$

$$\mathcal{H} = \frac{4q^2}{\sqrt{h}\phi^2}\pi^{ij}\pi_{ij} - \frac{2q^2}{\sqrt{h}\phi}\pi\pi_\phi - \frac{\sqrt{h}}{4q^2}\phi^2\, {}^{(3)}R + \frac{3q^2}{4\sqrt{h}}\pi_\phi^2 - \frac{\sqrt{h}}{q^2}\partial^i\phi\partial_i\phi + \frac{\sqrt{h}}{q^2}\nabla^i(\phi\phi_{,i}) + \frac{\sqrt{h}}{4q^2}U(\phi)$$

- An interesting result is the momentum constraints are the generetars of the space diffeomeorphism on the three-dimensional spatial surfaces

$$\{h_{ij}(x), \int d^3y N^l \mathcal{H}_l\} = \mathcal{L}_{\mathbf{N}} h_{ij}(x) \quad \{\pi^{ij}(x), \int d^3y N^l \mathcal{H}_l\} = \mathcal{L}_{\mathbf{N}} \pi^{ij}(x)$$

- From this relations, like in standard Hamiltonian General Relativity, follows

$$\{\mathcal{H}_i(x), \mathcal{H}_j(x')\} = \mathcal{H}_i(x')\partial_j\delta(x,x') - \mathcal{H}_j(x)\partial_i'\delta(x,x')$$

ADM ANALYSIS OF BRANS-DICKE THEORY

- The Poisson bracket Hamiltonian momentum constraints is

$$\{\mathcal{H}(x), \mathcal{H}_i(x')\} = -\mathcal{H}(x')\partial_i'\delta(x', x)$$

- While quite problematic (and under study…) is the Poisson bracket Hamiltonian-Hamiltonian constraint

$$\{N(x)\mathcal{H}(x), N(x')\mathcal{H}(x')\} = ?$$

- Still ongoing calculations. A first attempt seems to show the constraint algebra does not close. More later…

COSMOLOGIES OF THE SUB-PLANCK ERA

- Consider the previous E-H action with matter

$$S=\int_M d^4x\sqrt{-g}\left\{\frac{R-2\Lambda(k)}{16\pi G(k)}+\mathcal{L}_m\right\}+\frac{1}{8\pi}\int_{\partial M}\frac{K\sqrt{h}}{G(k)}d^3x$$

- One starts from a FLRW metric, in which the shifts $N^i = 0$

$$ds^2=-N(t)^2dt^2+\frac{a(t)^2}{1-Kr^2}dr^2+a(t)^2(r^2d\theta^2+r^2\sin\theta d\phi^2)$$

- Perfect fluid, with density ρ and pressure p and equation of state $p = w\rho$, w is a constant. Imposing the conservation of matter stress energy tensor $T^{\mu\nu}{}_{;\nu}=0$ one get $\rho(a)=ma^{-3-3w}$ with m an integration constant, and $\mathcal{L}_m=-mNa^{-3w}$

- Following Manrique et al. (ADM cut-off identification $k\sim\frac{1}{a}$)

$$\mathcal{L}_g=\frac{3\,a\dot{a}^2}{8\pi N(t)G(a)}+\frac{3\,a^2\dot{a}^2G'(a)}{8\pi NG^2(a)}+\frac{3\,aNK}{8\pi G(a)}-\frac{a^3N\Lambda(a)}{8\pi G(a)}-\frac{2Nm}{a^{3w}}$$

COSMOLOGIES OF THE SUB-PLANCK ERA

- From the Hamiltonian constraint, one gets the quantum-Friedman

$$\frac{K}{a^2H^2} - \frac{8\pi G(a)\,\rho + \Lambda(a)}{3H^2} + \eta(a) + 1 = 0$$

in which $\eta(a) = -\frac{\partial \log G(a)}{\partial \log a}$

- This implies an equation of evolution for $a(t)$

$$\dot{a}^2 = -\tilde{V}_K(a) \equiv -\frac{K + V(a)}{\eta(a) + 1} \qquad V(a) = \frac{a^2}{3}(8\pi G(a)\,\rho + \Lambda(a))$$

- Notice the allowed regions for the dynamical evolution are $\tilde{V}_K(a) \leq 0$.

- Close to NGFP, using cut off $k \sim \frac{1}{a}$, the follwing approximate solution for RG-equation are deduced (Biemans et al. 2017)

$$G(a) \approx G_0\left(1 + G_0/g_* a^{-2}\right)^{-1} \qquad (\lambda_*, g_*) \quad \text{NGFP,}$$

$$\Lambda(a) \approx \lambda_* a^{-2} + \lambda_0 \qquad (\lambda_0, G_0) \quad \text{IR valule}$$

BOUNCING AND EMERGENT UNIVERSES

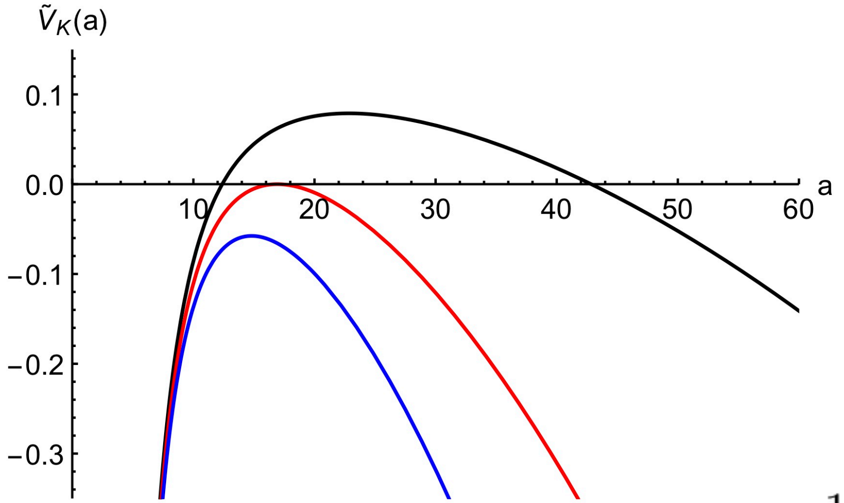

FIG. 1. The effective potential $\tilde{V}_K(a)$ for a bouncing universe (black), emergent universe (red), singular universe (blue), for $K = 0$, $w = 1/3$, $g_* = 0.1$, $\lambda_* = -0.5$, $\xi = 1$ and $m = 3$. Black, red and blue correspond to $\lambda_0 = 2 \times 10^{-4}$, $\lambda_0 = 8.3 \times 10^{-4}$ and $\lambda_0 = 1.5 \times 10^{-3}$ respectively.

- In the radiation dominated era $w = \frac{1}{3}$ $\tilde{V}_K(a) = 0$ has two solutions with non negative real part

$$a_b^2 = -\frac{G_0\Lambda_0 + g_*(\lambda_* - 3K)}{2g_*\Lambda_0} \pm \sqrt{\left(\frac{G_0\Lambda_0 - g_*(\lambda_* - 3K)}{2g_*\Lambda_0}\right)^2 - \frac{8\pi m G_0}{\Lambda_0}}$$

- The special condition of the emergent universe holds

$$\dot{a}_b = \ddot{a}_b = 0$$

- Condition for a_b^2 positive is $\lambda_* - 3K < -\frac{G_0\Lambda_0}{g_*}$, in the classical case $\lambda_* = 0$ and K>0 is the only possible case, in AS with matter K=-1 and K=0 are allowed as well.

EMERGENT UNIVERSES

- Around the minimal radius a_b, in the Emergent Universe case, one can linearize the first order differential equation in the following way

$$\dot{a}^2 = \frac{4g_* a_b^2 \Lambda_0}{3\left(g_* a_b^2 - G_0\right)} (a - a_b)^2$$

- The general solution is then

$$a(t) = a_b + \epsilon \exp\left\{ \sqrt{\frac{4g_* a_b^2 \Lambda_0}{3\left(g_* a_b^2 - G_0\right)}}\, t \right\}$$

where ϵ is an integration constant.

- There is an exponential evolution and then no *ad hoc* inflation. In particular it follows that the density parameter is

$$\Omega - 1 = \frac{3\left(g_* a_b^2 - G_0\right) K}{4g_* a_b^4 \Lambda_0}\, e^{-2N_e}$$

- The number N_e of e-folds, then, is related to the time t_e of exit from inflation by

$$N_e \simeq \log\left(\frac{\epsilon}{a_b} \exp\left\{ \sqrt{\frac{4g_* a_b^2 \Lambda_0}{3\left(g_* a_b^2 - G_0\right)}}\, t_e \right\} \right)$$

CONCLUSIONS

- Hamiltonian (ADM) analysis of RG improved Einstein-Hilbert action with G and Λ as external, non geometrical field, has been performed. The constraint algebra does not close in the general case. In the particular case of an ADM metric in Gaussian Normal coordinates, the constraint algebra do close without any restriction on the functional form of G=G(x) and $\Lambda(x)$.

- The very fact the algebra does not close "seems to be true also for Brans-Dicke theory", although more careful checks seem needed. Maybe the Hamiltonian formalism, in cases a bit more complicated than Einstein General Relativity, becomes, with its ADM-3+1 decomposition, too complicated.

- FLRW metrics have been studies in the minisupersapce approach using Dirac's constraint theory as a Hamiltonian cosmological applicacion of the above analysis. They generate sub-Planckian cosmological models via Asymptotic Safety. They exhibit Bouncing and Emergent Universes also in cases K=-1,0,1, that are impossible to draw from Classical General Relativity. Singularity is absent in the quantum regime!

- Minisuperspace models are more interesting to study. Therefore a further analysis is the study of an ADM Black Hole in the sub-Planckian regime.