Progress in the solving nonperturbative renormalization group for tensorial group field theory

Vincent Lahoche and Dine Ousmane Samary

CEA-LIST (France) and CIPMA-CHAIR-UNESCO (Benin)

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1. Introduction : motivations
2. Truncation method and flow equations
3. Effective vertex expansion
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Abstract

This presentation aims at giving our new advance on the functional renormalization group applied to tensorial group field theory. It is based on a serie of our three papers [arXiv :1803.09902], [arXiv :1809.00247] and [arXiv :1809.06081]. We consider the polynomial Abelian $U(1)^d$ models without closure constraint, especially we discuss the case of quartic melonic interaction. By using the effective vertex expansion method we studied the Wetterich flow equation and the possible existence or not of the phase transition in the leading order melonic sector. We provide the so called structure equations compactible with the Ward identities. These equations allow a new constraint on the flow i.e. on to the β-functions and the anomalous dimension η. We also show that, by adding in the functional renormalization group program other leading order contribution in the interaction, the physical conclusions such as the non-existence of fixed points for quartic model are not modified.

Introduction

Motivations

1. In the search of unify theory of modern physics, i.e. a well defined theory of quantum gravity, a lot of efforts have been made. Despite the fact that today, none of them may give entirely the complete definition on this issue, several major advances are observed to tackle this very important problem. Among which we can identify a very recent directions such as loop quantum gravity, dynamical triangulation, noncommutative geometry, group field theories (GFTs) and tensors models (TMs).
2. These approachs are considered as new background independent approaches according to several theoreticians. GFTs are quantum field theories over the group manifolds and are considered as the second quantization version of loop quantum gravity. These theories are caracterized by the specific form of non-locality in their interactions. TM, especially colored ones, allow one to define probability measures on simplicial pseudo-manifolds such that the tensor of rank d represents a $(d-1)$-simplex.

Introduction

Motivations

1. TMs admit the large N-limit dominated by the graphs called melons thanks to the Gurau breakthrough. This limit behaviour is a powerful tool which allows us to understand the continuous limit of these models through, for instance, the study of critical exponents and phase transitions.
2. TM and GFT are merged to give birth to a new class of field theories called tensorial group field theory. These class of field models enjoy renormalization and asymptotic freedom. Using the functional renormalization group (FRG) method, it is also possible to identify the equivalent of Wilson-Fisher fixed point for some particular cases of models and to prove the asymptotically freedom and safety.

Introduction

Wilson and Polchinski

1. There are several way to introduce the FRG in field theories. The first approach is the one pioneered by Wilson, which follows from simple and intuitive and therefore yieds a powerful way to think about quantum field theories. This method allows to interpolate smoothly between the known microscopic laws IR-regime and the complicated macroscopic phenomena in physical systems UV-regime and is constructed with the incomplete integration as cutoff procedure.
2. Well after a new approach to address the same question inspired from the Wilsonian method, called Wilson-Polchinski FRG equation is given. This very practicable method, may be integrated with an arbitrary cutoff function and expanded up to the next to leading order of the derivative expansion. Despite the fact that all these approachs seem to be nonperturbatively, in practice, the perturbative solution has appeared more attractive.

Introduction

Wetterich equation

1. After Wilson-Polchinski, the so called Wetterich flow equation is proposed to study the nonperturbative FRG and whose study requires approximations or truncations and numerically analysis due to the nonlinearity form.
2. The FRG equation allows to determine the fixed points and probably the phase transition. These phase transitions in the case of TGFT models may help to identify the emergence of general relativity and quantum mechanics through the geometrogenesis scenario.
3. In our recent and modest contributions the effective vertex expansion method is used in the context of the FRG. This leads to the definition of new class of equations called structure equations that help to solve the Wetterich flow equation. Taking into account the leading order contribution in the symmetric phase, the non-perturbative regime without truncation can be studied.

Introduction

Our contribution and main goal of this presentation

1. The Ward-Takahashi (WT) identities is derived and become a constraint along the flow. These identities are universal for all field theories having a symmetry, and are not specific to TGFT. Therefore all the fixed points must belong inside to the domain of this constraint line, before being considered as an acceptable fixed points.
2. In the case of quartic melonic TGFT models we have showed that the fixed point occurring from the solution of Wetterich equation violate this constraint for any choices of regulator functions. This violation is also independent of the method used to find this fixed point, whether it is the truncation, or the EVE method.
3. We will discuss all this point in the following presentation

The model

In the context of TGFT, we consider the pair of complex fields ϕ and $\bar{\phi}$ which takes values of d-copies of arbitrary group G :

$$\phi, \bar{\phi} : G^d \to \mathbb{C} \tag{1}$$

The particular case is $G = U(1)$ the Abelian compact Lie group. For the rest we consider only the Fourier transform of the fields ϕ and $\bar{\phi}$ denoted respectively by $T_{\vec{p}}$ and $\bar{T}_{\vec{p}}$, $\vec{p} \in \mathbb{Z}^d$ written as (for $\vec{g} \in U(1)^d$, $g_j = e^{i\theta_j}$) :

$$\phi(\vec{\theta}) = \sum_{\vec{p}\in\mathbb{Z}^d} T_{\vec{p}}\, e^{i\sum_{j=1}^d \theta_j p_j}, \quad \bar{\phi}(\vec{\theta}) = \sum_{\vec{p}\in\mathbb{Z}^d} \bar{T}_{\vec{p}}\, e^{-i\sum_{j=1}^d \theta_j p_j}. \tag{2}$$

The description of the statistical field theory is given by the partition function

$$\mathcal{Z}[J, \bar{J}] = \int d\mu_C\, e^{-S_{int} + \langle J, \bar{T}\rangle + \langle T, \bar{J}\rangle}, \tag{3}$$

The model

C is the covariance taking to be

$$C(\vec{p}) = \frac{1}{\vec{p}^{\,2} + m^2} = \int d\mu_C \, T_{\vec{p}} \, \bar{T}_{\vec{p}} \tag{4}$$

In order to prevent the UV divergences and supress the high momenta contributions, the propagator (4) has to be regularized. Schwinger regularization :

$$C_\Lambda(\vec{p}) = \frac{e^{-(\vec{p}^{\,2}+m^2)/\Lambda^2}}{\vec{p}^{\,2} + m^2}. \tag{5}$$

In general case, Let $\vartheta(t)$ such that $|1-\vartheta(t)| \leq Ce^{-\kappa t}$ for $C, \kappa > 0$ and $t \to +\infty$, a Laplace transform yields :

$$C_\Lambda(\vec{p}) = \int_0^{+\infty} dt \, \vartheta(t\Lambda^2) \, e^{-t(\vec{p}^{\,2}+m^2)}, \quad [\vartheta(t) = \Theta(t-1) \Rightarrow \text{Schwinger}] \tag{6}$$

The model

We introduce tensorial unitary invariants. An invariant is a polynomial $P(T, \bar{T})$ in the tensor entries $T_{\vec{p}}$ and $\bar{T}_{\vec{p}}$ which is invariant under the following action of $U(N)^{\otimes d}$ as follows :

$$T_{\vec{p}} \to \sum_{\vec{q}} U^{(1)}_{p_1 q_1} \cdots U^{(d)}_{p_d q_d} T_{\vec{q}}, \quad \bar{T}_{\vec{p}} \to \sum_{\vec{q}} \bar{U}^{(1)}_{p_1 q_1} \cdots \bar{U}^{(d)}_{p_d q_d} \bar{T}_{\vec{q}}. \tag{7}$$

The algebra of invariant polynomials is generated by a set of polynomials labeled by bubbles :

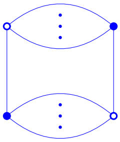
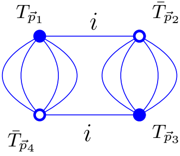
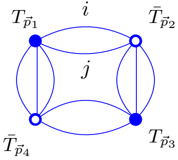


FIGURE – The 4-vertex bubble from which the dots indicate multiple edges.

The model

We consider the quartic melonic T_5^4 model which is proved to be renormalizable in all orders in the perturbative theory. The interaction of this model taking into account the leading order contributions : (melon and pseudo-melon) is written graphically as :

$$S^4_{int} = \lambda_{41} \sum_{i=1}^{5}$$ 

$$+ \lambda_{42} \sum_{i,j=1}^{5}$$ 

$$+ \cdots \quad (8)$$

$$S^4_{kin} = T_{\vec{p}}$$  $$\bar{T}_{\vec{p}} \quad (9)$$

Wetterich flow equation

The Wetterich equation is a functional integro-differential equation for the effective action Γ, now taking into account the quantum fluctuations caracterized by the parameter s, and called average effective action denoted by Γ_s, $-\infty < s < +\infty$. It is the Legendre transformation of the standard free energy $\mathcal{W}_s = \ln \mathcal{Z}_s$:

$$\Gamma_s[M, \bar{M}] = \langle \bar{J}, M \rangle + \langle \bar{M}, J \rangle - \mathcal{W}_s[J, \bar{J}] - R_s[M, \bar{M}] \tag{10}$$

where $R_s[M, \bar{M}] := Tr(M r_s \bar{M})$ and r_s is called the IR regulator. The appearance of this regulator r_s is introduced as new parameter function, which controls the scale fluctuation from IR to UV such that

$$\lim_{s \to -\infty} r_s = 0, \quad \lim_{s \to +\infty} r_s = \infty. \tag{11}$$

This definition ensures that Γ_s satisfies the boundary conditions $\Gamma_{s=\ln \Lambda} = S$, $\Gamma_{s=-\infty} = \Gamma$, where Λ is the UV cutoff.

Wetterich flow equation

The fields M and $\bar{M}$ are the mean values of T and $\bar{T}$ respectively and are given by

$$M = \frac{\partial \mathcal{W}}{\partial \bar{J}}, \quad \bar{M} = \frac{\partial \mathcal{W}}{\partial J} \tag{12}$$

where $\mathcal{W} := \mathcal{W}_{s=-\infty}$. In general the regulator r_s is chosen to be $r_s = Z(s) k^2 f\left(\frac{\vec{p}^{\,2}}{k^2}\right)$, $k = e^s$, and such that the conditions (11) is well satified. Let $\Gamma_s^{(2)}$ is the second order partial derivative of Γ_s with respect to the mean fields M and $\bar{M}$, the Wetterich equation is then given by

$$\partial_s \Gamma_s = Tr\, \partial_s r_s (\Gamma_s^{(2)} + r_s)^{-1} \tag{13}$$

The average effective action is chosen to be of the form

$$\Gamma_s = Z(s) \sum_{\vec{p} \in \mathbb{Z}^d} T_{\vec{p}} (\vec{p}^{\,2} + e^{2s} \bar{m}^2(s)) \bar{T}_{\vec{p}} + \sum_n Z(s)^{\frac{n}{2}} \bar{\lambda}_n V_n(T, \bar{T}) \tag{14}$$

Wetterich flow equation

In the case of quartic melonic interaction and by taking the standard modified Litim's regulator :

$$r_s(\vec{p}) = Z(s)(e^{2s} - \vec{p}^{\,2})\Theta(e^{2s} - \vec{p}^{\,2}) \tag{15}$$

the Wetterich equation can be solved analytically and the phase diagram may be given. The flow equations are

$$\begin{cases} \dot{m}^2 & = -2d\lambda I_2(0) \\ \dot{Z}(s) & = -2\lambda I_2'(q=0) \\ \dot{\lambda}_{41} & = 4\lambda_{41}^2 I_3(0) \end{cases} \qquad I_n(q) = \sum_{\vec{p}\in\mathbb{Z}^{(d-1)}} \frac{\dot{r}_s}{(Z(s)\vec{p}^{\,2} + Zq^2 + m^2 + r_s)^n}. \tag{16}$$

with the renormalization condition

$$m^2(s) = \Gamma_s^{(2)}(\vec{p} = \vec{0}), \quad \lambda_{41}(s) = \frac{1}{4}\Gamma_s^{(4)}(\vec{0},\vec{0},\vec{0},\vec{0}). \tag{17}$$

Wetterich flow equation

Explicitly using the integral representation of the above sum and with $d = 5$, $\eta = \dot{Z}/Z$ we get

$$I_n(0) = \frac{\pi^2 e^{6s-2ns}}{6Z(s)^{n-1}(\bar{m}^2+1)^n}(\eta+6), \quad I_n'(0) = -\frac{\pi^2 e^{4s-2ns}}{2Z(s)^{n-1}(\bar{m}^2+1)^n}(\eta+4). \quad (18)$$

In term of dimensionless parameter $\lambda_{41} = Z^2\bar{\lambda}_{41}$, $m^2 = e^{2s} Z \bar{m}^2$ the system (16) becomes

$$\begin{cases} \beta_m &= -(2+\eta)\bar{m}^2 - 2d\bar{\lambda}\,\frac{\pi^2}{(1+\bar{m}^2)^2}\,\left(1+\frac{\eta}{6}\right)\,, \\ \beta_{41} &= -2\eta\bar{\lambda} + 4\bar{\lambda}^2\,\frac{\pi^2}{(1+\bar{m}^2)^3}\,\left(1+\frac{\eta}{6}\right)\,, \end{cases} \quad (19)$$

where $\beta_m := \dot{\bar{m}}^2$, $\beta_{41} := \dot{\bar{\lambda}}$ and :

$$\eta := \frac{4\bar{\lambda}\pi^2}{(1+\bar{m}^2)^2 - \bar{\lambda}\pi^2}\,. \quad (20)$$

Wetterich flow equation

The solutions of the system (19) is given analytically :

$$p_{\pm} = \Big(\bar{m}_{\pm}^2 = -\frac{23 \mp \sqrt{34}}{33}, \bar{\lambda}_{41,\pm} = \frac{328 \mp 8\sqrt{34}}{11979\pi^2}\Big). \tag{21}$$

Numerically

$$p_{+} = (-0.52, 0.0028), \quad p_{-} = (-0.87, 0.0036). \tag{22}$$

Apart from the fact that we have a singularity line around the point $\bar{m}^2 = -1$ in the flow equation (16), another second singularity arise from the anomalous dimension denominator, and corresponds to a line of singularity, with equation :

$$\Omega(\bar{m}, \bar{\lambda}) := (\bar{m}^2 + 1)^2 - \pi^2 \bar{\lambda}_{41} = 0 \tag{23}$$

This line of singularity splits the two dimensional phase space of the truncated theory into two connected regions characterized by the sign of the function Ω.

Wetterich flow equation

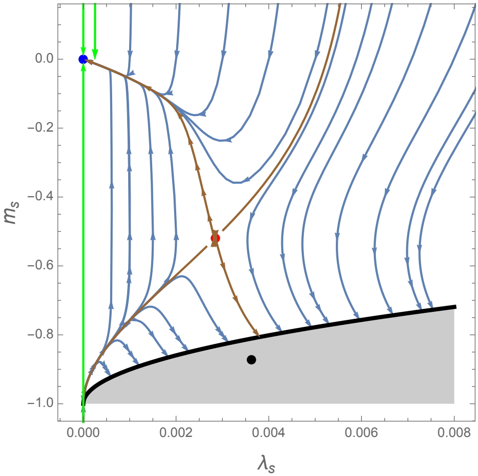
The region *I*, connected to the Gaussian fixed point for $\Omega > 0$ and the region *II* for $\Omega < 0$. For $\Omega = 0$, the flow becomes ill defined. The existence of this singularity is a common feature for expansions around vanishing means field, and the region *I* may be viewed as the domain of validity of the expansion in the symmetric phase. Note that to ensure the positivity of the effective action, the melonic coupling must be positive as well. Therefore, we expect that the physical region of the reduced phase space correspond to the region $\lambda_{41} \geq 0$. From definition of the connected region *I* and because of the explicit expression of anomalous dimension, we deduce that :

$$\eta \geq 0\,, \qquad \text{In the symmetric phase}\,. \tag{24}$$

Then, only the fixed point p_+ much be taken into account.

Phase diagram

The phase diagram is given :



Ward-identities

Let $\mathcal{U} = (U_1, U_2, \cdots, U_d)$, where the $U_i \in U_\infty$ are infinite size unitary martices in momentum representation. We define the transformation :

$$\mathcal{U}[T]_{\vec{p}} = \sum_{\vec{q}} U_{1\,,p_1q_1} U_{2\,,p_2q_2} \cdots U_{d\,,p_dq_d} T_{\vec{q}}\,, \tag{25}$$

such that the interaction term is invariant i.e. $\mathcal{U}[S_{int}] = S_{int}$. Then consider an infinitesimal transformation :

$$\mathcal{U} = \mathbf{I} + \vec{\epsilon}, \quad \vec{\epsilon} = \sum_i \mathbb{I}^{\otimes(i-1)} \otimes \epsilon_i \otimes \mathbb{I}^{\otimes(d-i)}\,, \tag{26}$$

where $\mathbb{I}$ is the identity on U_∞, $\mathbf{I} = \mathbb{I}^{\otimes d}$ the identity on $U_\infty^{\otimes d}$, and ϵ_i denotes skew-symmetric hermitian matrix such that $\epsilon_i = -\epsilon_i^\dagger$ and $\vec{\epsilon}_i[T]_{\vec{p}} = \epsilon_{i\,p_iq_i} T_{p_1,\cdots,q_i,\cdots,p_d}$.

Ward-identities

The invariance of the path integral (3) means $\vec{\epsilon}[\mathcal{Z}_s[J,\bar{J}]] = 0$, i.e. :

$$\vec{\epsilon}[\mathcal{Z}_s[J,\bar{J}]] = \int dTd\bar{T}\left[\vec{\epsilon}[S_{kin}] + \vec{\epsilon}[S_{int}] + \vec{\epsilon}[S_{source}]\right] e^{-S_s[T,\bar{T}]+\langle \bar{J},T\rangle + \langle \bar{T},J\rangle} = 0. \tag{27}$$

Computing each term separately, we get successively using linearity of the operator $\vec{\epsilon}$: $\vec{\epsilon}[S_{int}] = 0$, $\vec{\epsilon}[S_{source}] = -\sum_{i=1}^{d}\sum_{\vec{p},\vec{q}}\prod_{j\neq i}\delta_{p_j q_j}[\bar{J}_{\vec{p}}\,T_{\vec{q}} - \bar{T}_{\vec{p}}J_{\vec{q}}]\epsilon_{i\,p_i q_i}$, $\vec{\epsilon}[S_{kin}] = \sum_{i=1}^{d}\sum_{\vec{p},\vec{q}}\prod_{j\neq i}\delta_{p_j q_j}\,\bar{T}_{\vec{p}}\big[C_s(\vec{p}^{\,2}) - C_s(\vec{q}^{\,2})\big]\,T_{\vec{q}}\,\epsilon_{ip_iq_i}$. where $\prod_{j\neq i}\delta_{p_jq_j} := \delta_{\vec{p}_{\perp_i}\vec{q}_{\perp_i}}$, $p_{\perp_i} := \vec{p}\setminus\{p_i\}$, $C_s^{-1} = C_{-\infty}^{-1} + r_s$ and $C_{-\infty}^{-1} = Z_{-\infty}\vec{p}^{\,2} + m^2_{-\infty}$. $Z_{-\infty}$ is the renormalized wave function usualy denoted by Z.

Ward-identities

The ward identity gives relation between two and four point functions as :

$$\sum_{\vec{r}_{\perp_i}, \vec{s}_{\perp_i}} \delta_{\vec{r}_{\perp_i} \vec{s}_{\perp_i}} (C_s^{-1}(\vec{r}) - C_s^{-1}(\vec{s})) \langle T_{\vec{r}} \bar{T}_{\vec{s}} T_{\vec{p}} \bar{T}_{\vec{q}} \rangle = -\delta_{\vec{p}_{\perp_i} \vec{q}_{\perp_i}} (G_s(p) - G_s(q)) \delta_{r_i s_i}, \quad (28)$$

where, defined by $\Gamma_s^{(4)}$, the 1PI four point function, we get

$$\langle T_{\vec{r}} \bar{T}_{\vec{s}} T_{\vec{p}} \bar{T}_{\vec{q}} \rangle = \Gamma^{(4)}_{s, \vec{r}\vec{s}; \vec{p}\vec{q}} \Big(G_s(\vec{p}) G_s(\vec{q}) + \delta_{\vec{r}\vec{p}} \delta_{\vec{s}\vec{q}} \Big) G_s(\vec{r}) G_s(\vec{s}) \quad (29)$$

Ward-identities

The formal invariance of the path integral implies that the variations of these terms have to be compensate by a non trivial variation of the source terms.

$$\sum_{i=1}^{d} \sum_{\vec{p}_{\perp_i}, \vec{q}_{\perp_i}} \delta_{\vec{p}_{\perp_i} \vec{q}_{\perp_i}} \left[\frac{\partial}{\partial J_{\vec{p}}} \left[C_s(\vec{p}^{\,2}) - C_s(\vec{q}^{\,2}) \right] \frac{\partial}{\partial \bar{J}_{\vec{q}}} - \bar{J}_{\vec{p}} \frac{\partial}{\partial \bar{J}_{\vec{q}}} + J_{\vec{q}} \frac{\partial}{\partial J_{\vec{p}}} \right] e^{\mathcal{W}_s[J,\bar{J}]} = 0 \,, \tag{30}$$

Then after few simplification we come to

$$\sum_{\vec{r}_{\perp_1}} G_s^2(\vec{r}) \frac{dC_s^{-1}}{dr_1^2}(\vec{r}) \Gamma^{(4)}_{s,\vec{r},\vec{r},\vec{p},\vec{p}} = \frac{d}{dp_1^2} \left(C_\infty^{-1}(\vec{p}) - \Gamma_s^{(2)}(\vec{p}) \right) . \tag{31}$$

or

$$Z_{-\infty} \mathcal{L}_s := \sum_{\vec{p} \in \mathbb{Z}^d} \left(Z_{-\infty} + \frac{\partial r_s}{\partial p_1^2}(\vec{p}) \right) G_s^2(\vec{p}) \delta_{p_1 0} \,. \tag{32}$$

Structure equations

The Structure equations is the relations between correlation function and allows to establish a constraint between β-functions for mass, interactions couplings and wave function renormalization. These relations are obtained in the deep UV limit (i.e. in the domain $1 \ll e^s \ll \Lambda$) without any assumption about the β-functions and without any truncation of the effective action Γ_s. The only assumption concern the choice of the initial conditions, ensuring the perturbative consistency of the full partition function. The first structure equation concern the self energy (or 1PI 2-point functions). It takes place as the *closed equation* for self energy. [1] Let us summarize in the following proposition

1. The rank of the tensors is fixed to 5, and we denote it by d to clarify the proof(s).

Structure equations

In the melonic sector, the self energy $\Sigma_s(\vec{p}\,)$ is given by the *closed equation* which takes into account the effective coupling $\lambda_{41}(s)$ as :

$$-\Sigma_s(\vec{p}\,) = 2\lambda_{41}^r Z_\lambda \sum_{\vec{q}} \left(\sum_{i=1}^{d} \delta_{p_i q_i} \right) G_s(\vec{q}\,)\,. \tag{33}$$

In the same way, in the melonic sector, the perturbative zero-momenta 1PI four-point contribution $\Gamma^{(4),i}_{s,\vec{0}\vec{0};\vec{0}\vec{0}}$ is given by :

$$\Gamma^{(4),i}_{s,\vec{0}\vec{0};\vec{0}\vec{0}} = 2\pi_{00} = \frac{4Z_\lambda \lambda_{41}^r}{1 + 2\lambda_{41}^r Z_\lambda \mathcal{A}_s}\,, \tag{34}$$

where $\mathcal{A}_s$ is defined as :

$$\mathcal{A}_s = \sum_{\vec{p}_\perp} [G_s(\vec{p}_\perp)]^2\,,\; \vec{p}_\perp := (0, p_1, \cdots, p_d)\,, \tag{35}$$

$G_s(\vec{p})$ being the effective propagator : $G_s^{-1}(\vec{p}\,) = Z_{-\infty}\vec{p}^{\,2} + m^2 + r_s(\vec{p}\,) - \Sigma_s(\vec{p}\,)$.

Structure equations

In other words, we have an explicit expression for the effective coupling $\lambda_{41}(s) := \frac{1}{4}\Gamma^{(4),i}_{s,\vec{0}\vec{0};\vec{0}\vec{0}}$,

$$\lambda_{41}(s) = \frac{\lambda_{41}^r}{1 + 2\lambda_{41}^r \bar{\mathcal{A}}_s}\,, \tag{36}$$

from which we get

$$\partial_s \lambda_{41}(s) = -\frac{2(\lambda_{41}^r)^2 \dot{\mathcal{A}}_s}{(1 + 2\lambda_{41}^r \Delta\mathcal{A}_s)^2} = -2\lambda_{41}^2(s)\dot{\mathcal{A}}_s\,. \tag{37}$$

In the above relation we introduce the dot notation $\dot{\mathcal{A}}_s = \partial_s \mathcal{A}_s$

$$\mathcal{A}_s = \sum_{\vec{p}_\perp} \frac{1}{[\Gamma_s^{(2)}(\vec{p}_\perp) + r_s(\vec{p}_\perp)]^2}, \quad \dot{\mathcal{A}}_s = -2\sum_{\vec{p}_\perp} \frac{\dot{\Gamma}_s^{(2)}(\vec{p}_\perp) + \dot{r}_s(\vec{p}_\perp)}{[\Gamma_s^{(2)}(\vec{p}_\perp) + r_s(\vec{p}_\perp)]^3}. \tag{38}$$

Constraint equation

The constraint providing from the Ward identity, which relies the β-functions and the anomalous dimension is given by :

$$\beta_{41} = -\eta\bar{\lambda}_{41}\left(1 - \frac{\bar{\lambda}_{41}\pi^2}{(1+\bar{m}^2)^2}\right) + \frac{2\bar{\lambda}_{41}^2\pi^2}{(1+\bar{m}^2)^3}\beta_m \tag{39}$$

This relation need to be taking into account in the Wetterich flow equation and therefore in the search of fixed point. To prove this relation, let us consider the derivative of Z with respect to s using the structure equation

$$\dot{Z} = (Z_{-\infty} - 2\lambda_{41}Z_{-\infty}\mathcal{L}_s)\frac{\dot{\lambda}_{41}}{\lambda_{41}} - 2Z_{-\infty}\dot{\Delta}_s\,\lambda_{41}. \tag{40}$$

In the above relation we have used the decomposition of $\mathcal{L}_s = \mathcal{A}_s + \Delta_s$. Remark that the Ward identity can be written as $2\lambda_{41}\mathcal{L}_s = 1 - \bar{Z}$ where $\bar{Z} = Z/Z_{-\infty}$.

Constraint equation

Then (40) becomes :

$$\frac{\dot{Z}}{Z} = \frac{\dot{\lambda}_{41}}{\lambda_{41}} - 2\frac{Z_{-\infty}}{Z}\dot{\Delta}_s\lambda_{41}. \tag{41}$$

We now use the dimensionless quantities $\bar{m}$, $\bar{\lambda}_{41}$, $\bar{B}_s$ such that $\Delta_s = \frac{\bar{Z}}{Z^2}\bar{B}_s$ and reexpressing (41) as :

$$\beta_{41} = -\eta\bar{\lambda}_{41} + 2\bar{\lambda}_{41}(-\eta\bar{B}_s + \dot{\bar{B}}_s) \tag{42}$$

where $\bar{B}_s$ and $\dot{\bar{B}}_s$ much be simply compute using the integral representation of the sum. We come to :

$$\bar{B}_s = -\frac{\pi^2}{2(1+\bar{m}^2)^2}, \quad \dot{\bar{B}}_s = \frac{\pi^2\beta_m}{(1+\bar{m}^2)^3}, \tag{43}$$

Constraint equation

Let p is a arbitrary fixed point of the theory. We get $\beta_m(p) = 0 = \beta_{41}(p) = 0$. Then the constraint (39) implies that a the point p

$$\eta\bar{\lambda}_{41}\Big(1 - \frac{\bar{\lambda}_{41}\pi^2}{(1+\bar{m}^2)^2}\Big)(p) = 0. \quad (44)$$

The particular solution $\bar{\lambda}_{41} = 0$ correspond to the Gaussian fixed point. For $\bar{\lambda}_{41} \neq 0$ we have only

$$\eta = 0, \text{ or } \frac{\bar{\lambda}_{41}\pi^2}{(1+\bar{m}^2)^2} = 1 \quad (45)$$

It is clear that the fixed point $p_+ = (-0.55, 0.0025)$, $\eta \approx 0.7$ violate these constraints.

Constraint equation

Remark that other way to prove the violation of the Ward identity, is to extract the coupling λ_{41} in the constraint (39) and solve the flow of mass and coupling taking into account the assumption that the coupling is determine by the constraint equation. Now we get

$$\bar{\lambda}_{41}^{3} = 0 \text{ or } \bar{\lambda}_{41} = \frac{11(1+\bar{m}^2)^2}{5\pi^2}. \tag{46}$$

By replacing this solution $\bar{\lambda}_{41}^{3} = 0$ in the flow equations of mass and coupling (16) we get

$$\beta_m = -2\bar{m}^2, \quad \beta_{41} = 0. \tag{47}$$

Now setting $\beta_m = 0 = \beta_{41}$, only the Gaussian fixed point $(\bar{m}^* = 0, \bar{\lambda}_{41}^* = 0)$ survives. Also the last solution leads to

$$\beta_m = \frac{4}{9}(12m+11), \quad \beta_{41} = \frac{484(m+1)(15m+13)}{225\pi^2}. \tag{48}$$

Now setting $\beta_m = 0 = \beta_{41}$, no solution for the mass exist in this case.

Flow equations using the EVE

Let us consider the flow equation for $\dot{\Gamma}^{(2)}$, obtained from (13) deriving with respect to M and $\bar{M}$:

$$\dot{\Gamma}^{(2)}(\vec{p}) = -\sum_{\vec{q}} \Gamma^{(4)}_{\vec{p},\vec{p},\vec{q},\vec{q}} \, G_s^2(\vec{q}\,)\dot{r}_s(\vec{q}\,)\,, \tag{49}$$

where we discard all the odd contributions, vanishing in the symmetric phase. Deriving on both sides with respect to p_1^2, and setting $\vec{p} = \vec{0}$, we get :

$$\dot{Z} = -\sum_{\vec{q}} \Gamma^{(4)\,\prime}_{\vec{0},\vec{0},\vec{q},\vec{q}} \, G_s^2(\vec{q}\,)\dot{r}_s(\vec{q}\,) - \Gamma^{(4)}_{\vec{0},\vec{0},\vec{q},\vec{q}} \, G_s^2(\vec{q}\,)\dot{r}_s(\vec{q}\,)\,, \tag{50}$$

where the "prime" designates the partial derivative with respect to p_1^2. In the deep UV ($k \gg 1$) the argument used in the T^4-truncation to discard non-melonic contributions holds, and we keep only the melonic diagrams as well. Moreover, to capture the momentum dependence of the effective melonic vertex $\Gamma^{(4)}_{\text{melo}}$ and compute the derivative $\Gamma^{(4)\,\prime}_{\text{melo}\,,\vec{0},\vec{0},\vec{q},\vec{q}}$, the knowledge of π_{pp} is required. It can be deduced from the same strategy as for the derivation of the structure equation, up to the replacement :

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$$\mathcal{A}_s \to \mathcal{A}_s(p) := \sum_{\vec{p}\in\mathbb{Z}^d} G_s^2(\vec{p})\delta_{p_1 p}\,, \tag{51}$$

from which we get :

$$\pi_{pp} = \frac{2\lambda_r}{1+2\lambda_r\bar{\mathcal{A}}_s(p)}\,, \quad \bar{\mathcal{A}}_s(p) := \mathcal{A}_s(p) - \mathcal{A}_{-\infty}(0)\,. \tag{52}$$

The derivative with respect to p_1^2 may be easily performed, and from the renormalization condition (17), we obtain :

$$\pi'_{00} = -4\lambda^2(s)\,\mathcal{A}'_s\,, \tag{53}$$

and the leading order flow equation for $\dot{Z}$ becomes :

$$\dot{Z} = 4\lambda^2\mathcal{A}'_s(0)\,I_2(0) - 2\lambda I'_2(0)\,. \tag{54}$$

Flow equations using the EVE

As announced, a new term appears with respect to the truncated version (16), which contains a dependence on η and then move the critical line. The flow equation for mass may be obtained from (49) setting $\vec{p}=\vec{0}$ on both sides. Finally, the flow equation for the marginal coupling λ may be obtained from the equation (13) deriving it twice with respect to each means fields M and $\bar{M}$. As explained before, it involves $\Gamma^{(6)}_{\text{melo}}$ at leading order, and to close the hierarchy, we use the marginal coupling as a driving parameter, and express it in terms of $\Gamma^{(4)}_{\text{melo}}$ and $\Gamma^{(2)}_{\text{melo}}$ only. One again, $\Gamma^{(6)}_{\text{melo}}$ have to be split into d monocolored components $\Gamma^{(6)\,,i}_{\text{melo}}$:

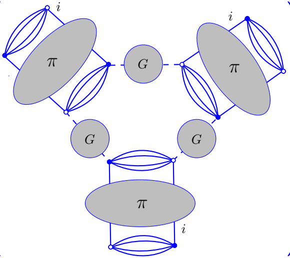
$$\Gamma^{(6)}_{\text{melo}} = \sum_{i=1}^{d} \Gamma^{(6)\,,i}_{\text{melo}} \,. \tag{55}$$

The structure equation for $\Gamma^{(6)\,,i}_{\text{melo}}$ may be deduced following the same strategy as for $\Gamma^{(4)\,,i}_{\text{melo}}$

Flow equations using the EVE

Starting from a vacuum diagram, a leading order 4-point graph may be obtained opening successively two internal tadpole edges, both on the boundary of a common internal face. This internal face corresponds, for the resulting 4-point diagram to the two external faces of the same colors running through the interior of the diagram. In the same way, a leading order 6-point graph may be obtained cutting another tadpole edge on this resulting graph, once again on the boundary of one of these two external faces. From this construction, it is not hard to see that the zero-momenta $\Gamma^{(6),i}_{\text{melo}}$ vertex function must have the following structure :

$$\Gamma^{(6),i}_{\text{melo}} = (3!)^2 \left(\cdots \right), \qquad (56)$$



Flow equations using the EVE

the combinatorial factor $(3!)^2$ coming from permutation of external edges. Translating the diagram into equation, and taking into account symmetry factors, we get :

$$\Gamma^{(6)\,,i}_{\text{melo}} = 24 Z^3(s)\bar{\lambda}^3(s) e^{-2s} \bar{\mathcal{A}}_{2s}\,, \tag{57}$$

with :

$$\bar{\mathcal{A}}_{2s} := Z^{-3} e^{2s} \sum_{\vec{p}\in\mathbb{Z}^{d-1}} G_s^3(\vec{p})\,. \tag{58}$$

Note that this structure equation may be deduced directly from Ward identities. The equation closing the hierarchy is then compatible with the constraint coming from unitary invariance. The flow equations involve now some new contributions depending on two sums, $\bar{\mathcal{A}}_{2s}$ and $\bar{\mathcal{A}}'_s$, defined without regulation function $\dot{r}_s$.

Flow equations using the EVE

However, they are both power-counting convergent in the UV, and the renormalizability theorem ensures their finitness for all orders in the perturbation theory. For this reason, they becomes independent from the initial conditions at scale Λ for $\Lambda \to \infty$; we get, using the Litim's regulator :

$$\bar{\mathcal{A}}_{2s} = \frac{1}{2}\frac{\pi^2}{1+\bar{m}^2}\left[\frac{1}{(1+\bar{m}^2)^2} + \left(1+\frac{1}{1+\bar{m}^2}\right)\right], \tag{59}$$

and

$$\bar{\mathcal{A}}'_s = \frac{1}{2}\pi^2\frac{1}{1+\bar{m}^2}\left(1+\frac{1}{1+\bar{m}^2}\right). \tag{60}$$

The complete flow equation for zero-momenta 4-point coupling write explicitly as :

$$\begin{aligned}\dot{\Gamma}^{(4)} &= -\sum_{\vec{p}} \dot{r}_s(\vec{p})G_s^2(\vec{p})\Big[\Gamma^{(6)}_{\vec{p},\vec{0},\vec{0},\vec{p},\vec{0},\vec{0}} \quad -2\sum_{\vec{p}\,'}\Gamma^{(4)}_{\vec{p},\vec{0},\vec{p}\,',\vec{0}}G_s(\vec{p}\,')\Gamma^{(4)}_{\vec{p}\,',\vec{0},\vec{p},\vec{0}} \\ &\quad +2G_s(\vec{p})[\Gamma^{(4)}_{\vec{p},\vec{0},\vec{p},\vec{0}}]^2\Big]. \end{aligned} \tag{61}$$

Flow equations using the EVE

Keeping only the melonic contributions, we get finally the following autonomous system by using the Litim's regulation :

$$\begin{cases} \beta_m &= -(2+\eta)\bar{m}^2 - 2d\bar{\lambda}\,\frac{\pi^2}{(1+\bar{m}^2)^2}\,\left(1+\frac{\eta}{6}\right)\,, \\ \beta_\lambda &= -2\eta\bar{\lambda} + 4\bar{\lambda}^2\,\frac{\pi^2}{(1+\bar{m}^2)^3}\,\left(1+\frac{\eta}{6}\right)\left[1-\frac{1}{2}\pi^2\bar{\lambda}\left(\frac{1}{(1+\bar{m}^2)^2}+\left(1+\frac{1}{1+\bar{m}^2}\right)\right)\right]. \end{cases} \tag{62}$$

where the anomalous dimension is then given by :

$$\eta = 4\bar{\lambda}\pi^2\frac{(1+\bar{m}^2)^2 - \frac{1}{2}\bar{\lambda}\pi^2(2+\bar{m}^2)}{(1+\bar{m}^2)^2\Omega(\bar{\lambda},\bar{m}^2) + \frac{(2+\bar{m}^2)}{3}\bar{\lambda}^2\pi^4}\,. \tag{63}$$

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The new anomalous dimension has two properties which distinguish him from its truncation version. First of all, as announced, the singularity line $\Omega = 0$ moves toward the $\bar{\lambda}$ axis, extending the symmetric phase domain. In fact, the improvement is *maximal*, the critical line being deported under the singularity line $\bar{m}^2 = -1$. In standard interpretations, the presence of the region *II* is generally assumed to come from a bad expansion of the effective average action around vanishing means field, becoming a spurious vacuum in this region. However, the EVE method show that this singularity line is completely discarded taking into account the momentum dependence of the effective vertex. The second improvement come from the fact that the anomalous dimension may be negative, and vanish on the line of equation $L(\bar{\lambda}, \bar{m}^2) = 0$, with :

$$L(\bar{\lambda}, \bar{m}^2) := (1 + \bar{m}^2)^2 - \frac{1}{2}\bar{\lambda}\pi^2(2 + \bar{m}^2)\,. \tag{64}$$

Interestingly, there are now two lines in the maximally extended region I' where physical fixed points are expected. However, numerical integrations, show that the improved flow equations admit a non-Gaussian fixed point, numerically very close from the fixed point p_+ obtained in the truncation method, and then unphysical.

Violation of Ward-identity

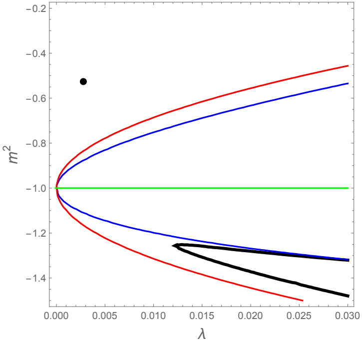

FIGURE – The relevant lines over the maximally extended region I', bounded at the bottom with the singularity line $m^2 = -1$ (in green). The blue and red curves correspond respectively to the equations $L = 0$ and $\Omega = 0$. Moreover, the black point correspond to the numerical non-Gaussian fixed point, so far from the two previous physical curves.

To sum up, in this work

- In this presentation we show that the IR fixed point obtained in the FRG applications for TGFT lack an important constraint coming from Ward identities. This constraint reduces the physical region of the phase space to a one-dimensional subspace without fixed point, suggesting that the phase transition scenario abundantly cited in the TGFT literature may be an artifact of an incomplete method.
- This suggestion is improved with a more sophisticated method, taking into account the momentum dependence of the effective vertex, and providing a maximal extension of the symmetric region. Despite with this improvement, the resulting numerical fixed point does not cross any of the physical lines provided from the Ward constraint. In the literature, the quartic truncation has been largely investigated, for various group manifold and dimensions.
- We expect from our analysis that none of these models modify our conclusions, except possibly for TGFT including *closure constraint* as a Gauge symmetry.

Thank you for your attention