Trace anomaly and infrared cutoffs

International Seminar on Asymptotic Safety 4/3/19

Tim Morris

TRM & Roberto Percacci 1810.09824

A theory that is scale invariant at the classical level, is in general no longer scale invariant at the quantum level.

This is known as the trace anomaly.

$$\delta_\epsilon g_{\mu\nu} = 2\epsilon g_{\mu\nu}$$

$$\delta_\epsilon \phi = -\epsilon\phi$$

(by length, 4 dimensions)

It results in the insertion in renormalised correlation functions of the anomalous trace of the renormalised stress-energy tensor:

$$\int d^4x\sqrt{g}$$

$$-\delta_\epsilon S = \epsilon \int_x T^\mu{}_\mu$$

Contributions from external non-flat metric & interactions.

We will focus on interactions.

$(g_{\mu\nu} = \delta_{\mu\nu})$

$\mu \partial_\mu \lambda(\mu)$

$\left(\frac{\lambda}{4!}\phi^4\right)$

$$\mathcal{A}(\epsilon) = -\delta_\epsilon S = \epsilon \int_x T^\mu{}_\mu = \epsilon\, \beta \int_x \frac{1}{4!}\phi^4$$

At the quantum level scale invariance is broken by the regularisation ⇨ breaking terms in bare action.

What meaning can scale invariance have now?

An `almost scale invariant' quantum theory.

$(g_{\mu\nu} = \delta_{\mu\nu})$

$\mu\partial_\mu\lambda(\mu)$

$\left(\frac{\lambda}{4!}\phi^4\right)$

$$\mathcal{A}(\epsilon) = -\delta_\epsilon S = \epsilon \int_x T^\mu{}_\mu = \epsilon\,\beta \int_x \frac{1}{4!}\phi^4$$

At the quantum level scale invariance is broken by the regularisation ⇨ breaking terms in bare action.

The breaking in the bare action is only by λ dependent contributions such that this equation is satisfied at the renormalised level.

Insertion of:

$$\mathcal{A}(\epsilon) = -\delta_\epsilon S = \epsilon \int_x T^\mu{}_\mu = \epsilon\, \beta \int_x \frac{1}{4!}\phi^4$$

$$\Downarrow$$

$$\delta_\epsilon \Gamma = -\mathcal{A}(\epsilon) = -\epsilon\, \beta(\lambda)\, \partial_\lambda \Gamma$$

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E.g. let $\langle\phi\rangle$ be the sole reason for breaking invariance:

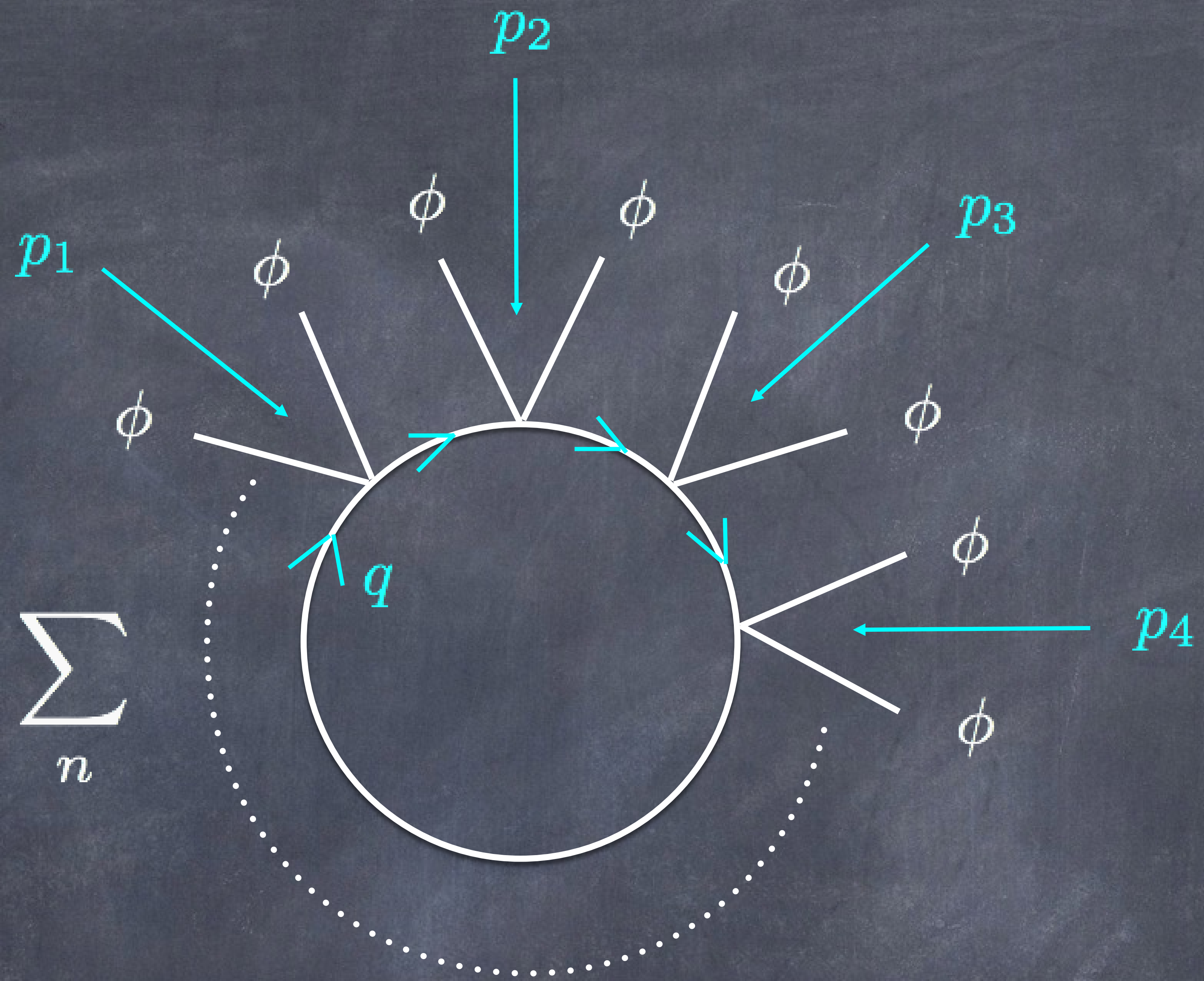
$$\Gamma = \int_x \frac{\langle\phi\rangle^4}{4!} v(\langle\phi\rangle)$$

$$\delta_\epsilon v(\langle\phi\rangle) = -\epsilon\, \langle\phi\rangle \partial_{\langle\phi\rangle} v = -\epsilon\, \beta(\lambda)\, \partial_\lambda v$$

$$\Gamma = \int_x \frac{\langle\phi\rangle^4}{4!} \lambda(\langle\phi\rangle) = \int_x \frac{\langle\phi\rangle^4}{4!} \left(\lambda(\mu) + \frac{3\lambda^2(\mu)}{(4\pi)^2} \log(\langle\phi\rangle/\mu) \right) .$$

Coleman-Weinberg potential (at one loop)

One loop:



bare action

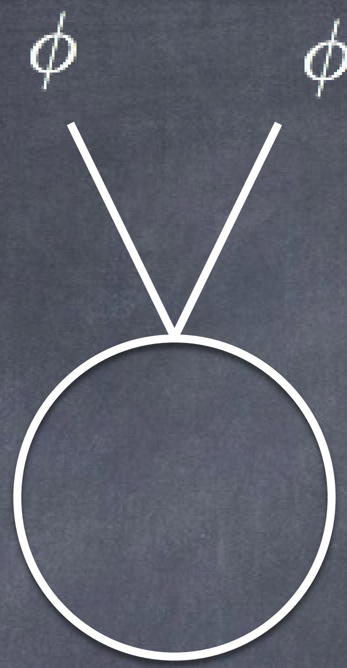
$$\Gamma[\phi] = S[\phi] + \sum_n -\frac{1}{2n}\left(-\frac{\lambda}{2}\right)^n \int^\Lambda \frac{d^4q}{(2\pi)^4}\,\frac{1}{q^2(q+P_1)^2(q+P_2)^2\cdots(q+P_{n-1})^2}$$

$$P_j = \sum_{i=1}^{j} p_j$$

$$\delta_\epsilon q^2 = -2\epsilon q^2$$

n>2: finite scale invariant non-local vertices n = 1

$$\frac{\lambda}{4}\int^{\Lambda}\frac{d^4q}{(2\pi)^4}\frac{1}{q^2}$$



gives a term in Γ which breaks scale invariance:

$$\Gamma \ni a\hbar\lambda\Lambda^2\int_x \phi^2$$

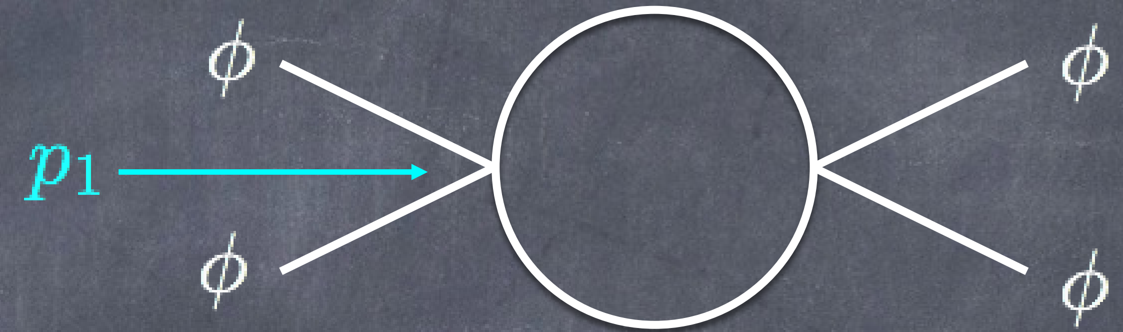
which we must cancel exactly with a counter-term that also breaks scale invariance:

$$S \ni -a\hbar\lambda\Lambda^2\int_x \phi^2$$

in order to restore scale invariance at the renormalised level

n = 2

$$A(p_1) = -\frac{\lambda^2}{16}\int^{\Lambda}\frac{d^4q}{(2\pi)^4}\frac{1}{q^2(q+p_1)^2}$$



Formally scale invariant but broken by UV regularisation:

$$\delta_\epsilon A(p_1) = -\frac{\lambda^2}{16}(+\epsilon\Lambda)\frac{2\Lambda^3}{(4\pi)^2}\frac{1}{\Lambda^4} = -\epsilon\,\frac{\beta(\lambda)}{4!}$$

Requires counterterm:

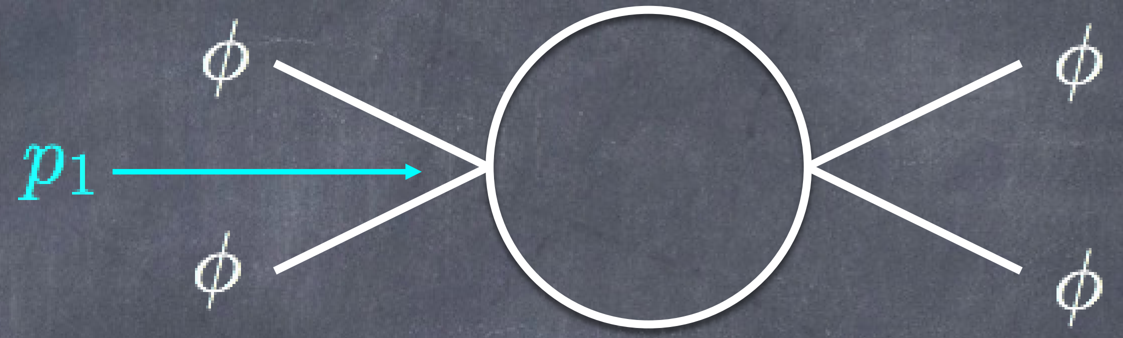
$$S \ni \int_x \left(\lambda(\mu) + \frac{3\lambda^2(\mu)}{(4\pi)^2}\log(\Lambda/\mu)\right)\frac{\phi^4}{4!}$$

$\lambda(\Lambda)$

which is scale invariant!

n = 2

$$A(p_1) = -\frac{\lambda^2}{16}\int^{\Lambda}\frac{d^4q}{(2\pi)^4}\frac{1}{q^2(q+p_1)^2}$$



With counter-term added, Λ has gone & vertex is finite, but scale invariance is still anomalous …

… because vertex has a particular non-local part:

$$\Gamma \ni \int_x \left\{ \frac{\lambda(\mu)}{4!}\phi^4 + \frac{\lambda^2(\mu)}{256\pi^2}\phi^2 \log\left(\frac{-\partial^2}{\mu^2}\right)\phi^2 \right\}$$

$$\delta_\epsilon\Delta = \delta_\epsilon(-\partial^2) = -2\epsilon\Delta$$

$$\delta_\epsilon\Gamma = -\epsilon\,\beta\int_x \frac{\phi^4}{4!}$$

$$\delta_\epsilon \Gamma = -\mathcal{A}(\epsilon)$$

Effective average action

$$\delta_\epsilon \Gamma_k = -\mathcal{A}(\epsilon) + \epsilon \partial_t \Gamma_k$$

$$S_k(\phi; g_{\mu\nu}) = \frac{1}{2}\int_x \phi R_k(\Delta)\phi$$

$$R_k(\Delta) = k^2 r(\Delta/k^2)$$

$$\delta_\epsilon R_k = \epsilon(-2R_k + \partial_t R_k)$$

$$\delta_\epsilon \Delta = \delta_\epsilon(-\partial^2) = -2\epsilon\Delta$$

Wilsonian RG!

$$\mathcal{A}(\epsilon) = \epsilon \partial_t \Gamma_k - \delta_\epsilon \Gamma_k$$

Fixed points $\Longrightarrow$ $\mathcal{A}(\epsilon) = 0$

Wilsonian RG

$$\mathcal{A}(\epsilon) = \epsilon \partial_t \Gamma_k - \delta_\epsilon \Gamma_k$$

Expand in local operators: $\Gamma_k = \sum_i \lambda_i(k)\, \mathcal{O}_i$

$$\tilde{\lambda}_i = k^{\Delta_i} \lambda_i$$

$$\tilde{\mathcal{O}}_i = k^{-\Delta_i} \mathcal{O}_i$$

Dimensionless vars: $\Gamma_k = \sum_i \tilde{\lambda}_i(k)\, \tilde{\mathcal{O}}_i$

$$\Longrightarrow \quad \mathcal{A}(\epsilon) = \epsilon \sum_i \tilde{\beta}_i\, \tilde{\mathcal{O}}_i$$

Almost scale invariance: $\tilde{\beta}_i = \partial_t \tilde{\lambda}_i = \partial_t \lambda \partial_\lambda \tilde{\lambda}_i = \beta(\lambda)\, \partial_\lambda \tilde{\lambda}_i$

$$\Longrightarrow \quad \mathcal{A}(\epsilon) = \epsilon\, \beta(\lambda)\, \partial_\lambda \Gamma_k$$

$$\delta_\epsilon \Gamma_k = -\mathcal{A}(\epsilon) + \epsilon \partial_t \Gamma_k$$

$$-\epsilon\, \beta(\lambda)\, \partial_\lambda \Gamma_k$$

E.g. let $\langle\phi\rangle$ be the sole reason for breaking invariance:

other

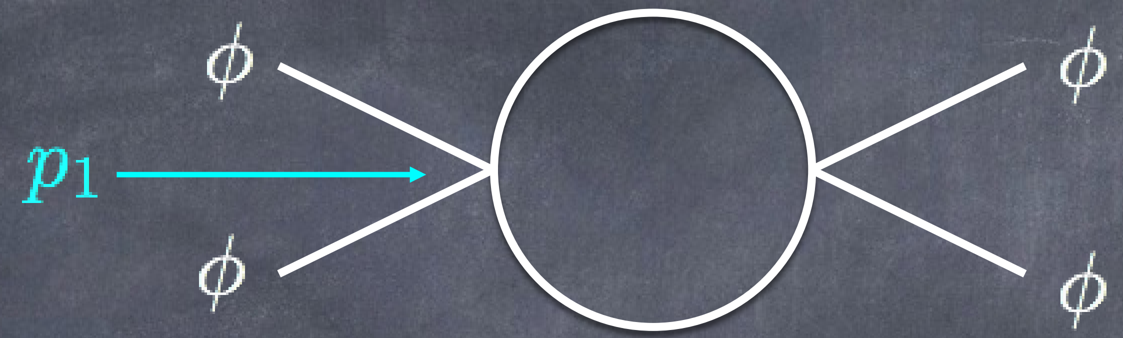
$$\Gamma_k = \int_x \frac{\langle\phi\rangle^4}{4!} v_k(\langle\phi\rangle)$$

$$\delta_\epsilon v_k(\langle\phi\rangle) = -\epsilon\, \langle\phi\rangle \partial_{\langle\phi\rangle} v_k = -\epsilon\, \beta(\lambda)\, \partial_\lambda v_k + \epsilon\, \partial_t\, v_k$$

$$v_k = v\left(\langle\phi\rangle/k\,,\ \lambda\big(\langle\phi\rangle^a k^{1-a}\big)\right)$$

(a any number)

n = 2



$$A_k(p_1) = -\frac{\lambda^2}{16}\int^{\Lambda}\frac{d^4q}{(2\pi)^4}\frac{1}{[q^2+R_k(q)]\,[(q+p_1)^2+R_k(q+p_1)]}$$

Derivative expansion:

$$\Gamma_k \;\ni\; \int_x \left\{ \frac{\lambda(k)}{4!}\,\phi^4 + \frac{\lambda^2(k)}{256\pi^2}\sum_{n=1}^{\infty} a_n\;\phi^2\left(\frac{-\partial^2}{k^2}\right)^n \phi^2 \right\}$$

$$\Gamma_k \ni \int_x \left\{ \frac{\lambda(k)}{4!} \phi^4 + \frac{\lambda^2(k)}{256\pi^2} \sum_{n=1}^{\infty} a_n \, \phi^2 \left(\frac{-\partial^2}{k^2} \right)^n \phi^2 \right\}$$

$$\delta_\epsilon \Gamma_k = -\mathcal{A}(\epsilon) + \epsilon \partial_t \Gamma_k$$

scale invariant!

$$\Downarrow$$

$$-\epsilon\, \beta(\lambda)\, \partial_\lambda \Gamma_k$$

$$-\int_x \frac{\lambda^2(k)}{128\pi^2} \sum_{n=1}^{\infty} n a_n \, \phi^2 \left(\frac{-\partial^2}{k^2} \right)^n \phi^2$$

$$\delta_\epsilon \int_k^\Lambda \frac{d^4 q}{(2\pi)^4} \frac{1}{q^4} = 0$$

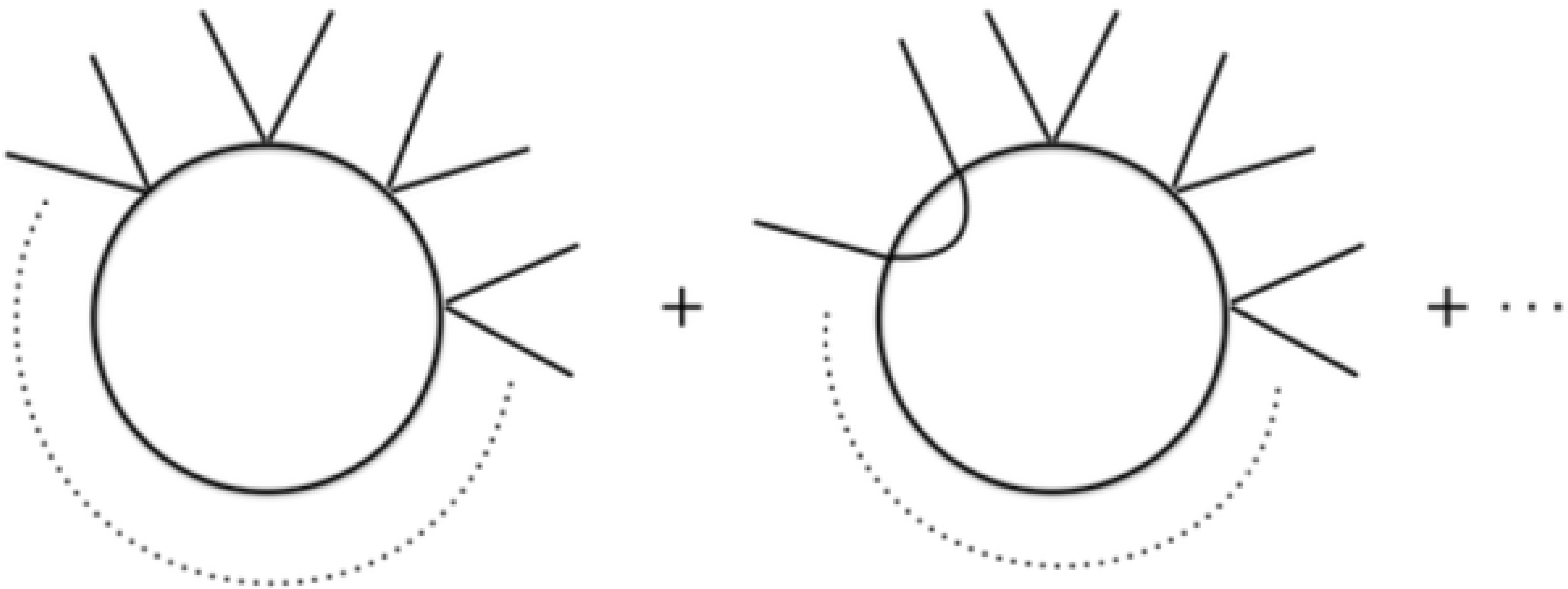


Figure 2: How higher-point vertices contribute to the trace anomaly beyond one loop.

Limit as k → 0

$$\Gamma_k = \int_x \left\{ \frac{1}{2}(\partial\phi)^2 + \frac{\lambda(k)}{4!}\,\phi^4 + \frac{\lambda^2(k)}{256\pi^2}\,\phi^2 \log\left(\frac{-\partial^2}{k^2}\right)\phi^2 + O(k^2\log k) \right\}$$

explicit k dependence diverges, but actually k independent! (RG invariant)

$$\delta_\epsilon \Gamma_k = -\mathcal{A}(\epsilon) + \epsilon\partial_t\Gamma_k$$

$$\Downarrow$$

$$\delta_\epsilon \Gamma = -\mathcal{A}(\epsilon) = -\epsilon\,\beta(\lambda)\,\partial_\lambda\Gamma$$

Trace anomaly and infrared cutoffs

- Trace anomaly defines `almost scale invariant'
- can be seen through influence of UV cutoff
- modified by IR cutoff, intimately related to RG
- Standard form hidden in local approximations
- … is recovered in the limit $k \to 0$, related to RG invariance.