Convergence of the derivative expansion and results at order ∂^6

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Why the renormalization group?

Consider a system with **many** degrees of freedom and a physical quantity defined at **large scale**:

Example: Ising (or ϕ^4) model: phase diagram, magnetization ($\langle\phi(x)\rangle$), correlation length (inverse mass), specific heat, ...

Typically, this quantity involves the sum of many microscopic degrees of freedom and gets contributions from fluctuations on all scales
$\longrightarrow$ if the central limit theorem holds, then, independently of the underlying theory, it is gaussian distributed: trivial case.

$\longrightarrow$ if the central limit theorem does not hold then the distribution is not gaussian and the RG is necessary.

Remark: This has nothing to do with the strength of the interaction among microscopic d.o.f. but with the strength of their correlations.

Why the nonperturbative (or functional or exact) renormalization group?

Renormalization group = machinery to build nongaussian (functional) probability measure.

Two possibilities
Either the probability measure is
$\longrightarrow$ closed to the gaussian, that is, the effective long-distance coupling between infrared d.o.f. is small and mean-field + perturbation theory is OK: ϕ^4 in $d = 4 - \epsilon$ dimensions or $N \to \infty$,
$\longrightarrow$ far from the gaussian and nonperturbative RG is necessary (when there is no exact solution) because it is not based on the smallness of a coupling constant.

Solution: Wilson's RG with nonperturbative approximation schemes.

The different NPRG schemes

Historically: two main implementations of approximate RG flows:

- Wilson-Polchinski: flows of hamiltonians or (almost equivalent) of Helmoltz free energies: $\mathcal{W}[J] = \log \mathcal{Z}[J] =$ generating functional of connected correlation functions,
- Wetterich (and Ellwanger and Morris and Parola-Reatto): flow of Gibbs free energies = Legendre transform of $\mathcal{W}[J]$= generating functional of 1PI correlation functions.

In principle, Wilson-Polchinski $\Leftrightarrow$ Wetterich.

When approximations are performed: Wetterich much better than Wilson-Polchinski: we shall see why.

Wilson's RG

- Statistical or quantum system given by:

$$\mathcal{Z}[J] = \int D\varphi \ \exp\Big\{ - H[\varphi] + \int_x J(x)\varphi(x)\Big\}$$

and supposed to be regularized at short distance:
Ising (lattice spacing a) or ϕ^4 with a UV cutoff $\Lambda \sim a^{-1}$.

> We will be interested in scale invariant systems
> $\Downarrow$
> close to criticality
> $\Downarrow$
> correlation length $= \xi \gg a \sim \Lambda^{-1} \Rightarrow m_R \ll \Lambda$

- $\mathcal{W}[J] = \ln \mathcal{Z}[J]$ (Helmoltz)
- $\phi(\vec{x}) = \langle \varphi(\vec{x}) \rangle = \frac{\delta W[J]}{\delta J(\vec{x})}$
- $\Gamma[\phi] + \mathcal{W}[J] = \int_x J_x \phi_x$ (Gibbs = Legendre transform)

Solving the theory = computing $\mathcal{Z}[J]$ (or $\Gamma[\phi]$)

$\Downarrow$

Integrate over all scales

- Wilson: summation over fluctuations momentum scale by momentum scale,
- $\Leftrightarrow$ "Block-spin" summation *à la* Kadanoff,
- summation over rapid modes (i.e. short distance or high energy) in order to:
  - Wilson-Polchinski: obtain an effective hamiltonian or (similarly) Helmoltz free-energy for the slow modes (not yet summed over);
  - Wetterich: obtain the effective (coarse-grained) Gibbs free energy corresponding to the integration over the rapid modes (already integrated over).
- transform the problem into a differential problem (important for approximations).

build a one-parameter-family of models

indexed by a scale k

that interpolates between H and Γ

$\Downarrow$

Integrate over the rapid modes only

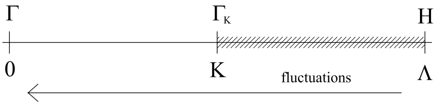
$\Downarrow$

freeze out the slow modes

by making them non-critical

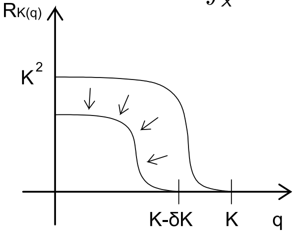
$\Downarrow$

Deduce the model with scale $k - \delta k$ from k only



- Concretely, perform the deformation:

$$\mathcal{Z}[J] \to \mathcal{Z}_k[J] = \int D\varphi \; \exp\left\{ -H[\varphi] - \Delta H_k[\varphi] + \int_x J(x)\varphi(x)\right\}$$



with

$$\Delta H_k[\varphi] = \frac{1}{2}\int_q R_k(q)\,\varphi_q\varphi_{-q}$$

must be such that

- when $k = \Lambda$: $\forall q, \quad R_{k=\Lambda}(q) \sim \Lambda^2 \Rightarrow$ all fluctuations are frozen $\Rightarrow$ mean-field approx. becomes exact: $\Rightarrow \quad \Gamma^{\mathrm{Leg}}_{k=\Lambda} = H + \Delta H_{k=\Lambda}$

$\Rightarrow \quad$ better to work with $\Gamma_k[\phi] = \Gamma_k^{\mathrm{Leg}}[\phi] - \Delta H_k[\phi]$

$\Rightarrow \Gamma_{k=\Lambda}[\phi] = H[\phi]$

- when $k = 0$: $\forall q, \quad R_{k=0}(q) = 0 \Rightarrow$ the original model is retrieved:

$\Rightarrow \quad \mathcal{Z}_{k=0}[J] = \mathcal{Z}[J]$ and $\Gamma_{k=0}[\phi] = \Gamma[\phi]$

- Exact flow of Γ_k [C.Wetterich, Phys.Lett B301 (1993) 90.]:

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \int_{x,y} \partial_t R_k(x-y) \left(\Gamma_k^{(2)} + R_k\right)^{-1}_{x,y}$$

where $\partial_t = k\partial_k$ and $(\Gamma_k^{(2)}[q;\phi(x)] + R_k(q))^{-1} = G_k[q;\phi(x)]$ is the "full" (field-dependent) propagator

- Some properties of Wetterich equation:
  - it is a differential equation in k ($\Gamma_{k-\delta k}$ is fixed from Γ_k only);
  - it involves only one loop integral (one loop structure);
  - allows for possible nonperturbative approximations;
  - the initial condition is the "bare" (microscopic) theory.
- Exact solution of Wetterich equation $\Leftrightarrow$ exact solution of the theory $\Rightarrow$ very different from Gell-Mann – Low RG!

- Flow equations for $\Gamma_k^{(n)}(p_i, \phi_{\text{unif.}})$ obtained by taking functional derivatives:

$$\partial_t \Gamma_k^{(2)}(p, \phi) = \int_q \partial_t R_k(q) G^2(q, \phi) \cdot$$

$$\left\{ \Gamma_k^{(3)}(p, q, \phi) G(p+q, \phi) \Gamma_k^{(3)}(-p, -q, \phi) - \frac{1}{2} \Gamma_k^{(4)}(p, -p, q, \phi) \right\}$$

- Difficulty: The flow equation of $\Gamma_k^{(n)}$ involves $\Gamma_k^{(n+1)}$ and $\Gamma_k^{(n+2)}$ $\Rightarrow$ infinite hierarchy of equations (not closed),
- A closure scheme is needed,
- Main point: Wetterich's formulation of Wilson RG is convenient for implementing nonperturbative approximation schemes.

What is the Derivative Expansion?

The most popular approximation scheme = Derivative expansion.

Usually, it consists in:
1. proposing an ansatz for $\Gamma_k[\phi]$ under the form of a gradient expansion. For the ϕ^4 theory at order ∂^2:

$$\Gamma_k[\phi] = \int_x \left\{ U_k(\phi(x)) + \frac{1}{2} Z_k(\phi(x))(\partial_\mu \phi)^2 + O(\partial^4) \right\}$$

2. computing $\Gamma_k^{(2)}, \cdots, \Gamma_k^{(n)}$ from this ansatz,
3. computing the propagator $\left(\Gamma_k^{(2)}(p,\phi) + R_k(p)\right)^{-1}$ from this ansatz,
4. plugging these expressions into the exact flow equations of Γ_k, $\Gamma_k^{(2)}$, $\cdots$,
5. projecting these flows onto the same functional subspace as the one chosen in the ansatz.

This procedure yields the flows of $U_k(\phi)$, $Z_k(\phi)$, $\cdots$

Is the (usual) DE an (Taylor) expansion?

NO!

If it were, we would consistently truncate the rhs of all the flow equations at the same order of the DE as Γ_k!

BUT, in the flow of $\Gamma_k^{(2)}$ (for instance):

$$\partial_t \Gamma_k^{(2)}(p,\phi) = \int_q \partial_t R_k(q) G^2(q,\phi)\cdot$$

$$\left\{\Gamma_k^{(3)}(p,q,\phi) G(p+q,\phi) \Gamma_k^{(3)}(-p,-q,\phi) - \frac{1}{2}\Gamma_k^{(4)}(p,-p,q,\phi\right\}$$

and at order 2 of the DE: $\Gamma_k^{(3)}(p,q,\phi)\Gamma_k^{(3)}(-p,-q,\phi)$ is of order ∂^4 and usually all terms of order 4 are kept while some other terms of order 4 have been neglected in the ansatz!

In the usual DE the ansatz of Γ_k is more considered as a book-keeping for the terms considered in the approximation than as a serious Taylor expansion!

What is the expansion parameter of the DE?

Intuitive argument:

The DE at order s consists in truncating all $\Gamma_k^{(n)}(\{p_i\},\phi)$ in their expansion in $p_i \cdot p_j$ at order s in the momenta $\Rightarrow$ it must be valid at small momenta.

But small momenta with respect to which scale?

1. Suppose we consider a critical (massless) theory: the only scale available is k: the expansion must be in $p_i \cdot p_j/k^2$, so $p_i \ll k$.

2. For a non critical (massive) theory:
- when $k \gg p_i, m$, same as before because the mass is negligible,
- when $m \gg k \gg p_i$, the flow almost stops because the regulator is negligible.

Expansion in:

$$\frac{p_i \cdot p_j}{k^2 + m^2} \quad ?$$

BUT two kinds of momenta in the DE:

- the external momenta, e.g. p in the flow of $\Gamma_k^{(2)}(p,\phi)$,
- the internal momentum q in

$$\partial_t \Gamma_k^{(2)}(p,\phi) = \int_q \partial_t R_k(q) G^2(q,\phi) \cdot$$

$$\left\{ \Gamma_k^{(3)}(p,q,\phi) G(p+q,\phi) \Gamma_k^{(3)}(-p,-q,\phi) - \frac{1}{2}\Gamma_k^{(4)}(p,-p,q,\phi) \right\}.$$

It is clear that p must be $\ll k$ for the DE to be valid and we have to restrict ourselves to the $p_i = 0$ region when using the DE...

BUT we cannot choose on which range varies q!
q varies between 0 and ∞.

Good news: the effective range of the integral over q in the flow is given by $\partial_t R_k(q)$: for a "good" regulator, it is effectively $[0, a\, q]$ with $a \simeq 1$.

To summarize: DE = expansion in q^2/k^2 (massless case); it needs to be convergent in the range $[0, k]$.

What do we know about the convergence of the DE?

In the massive case (at $k = 0$):

$$\frac{\Gamma^{(2)}(p)}{\Gamma^{(2)}(0)} = 1 + \frac{p^2}{m^2} + \sum_{n=2}^{\infty} c_n \left(\frac{p^2}{m^2}\right)^n$$

The c_n are universal in the critical regime $k \ll \Lambda$ and are known in the symmetric and broken phases!

coefficient	$d = 3,\ T > T_c$ HT / ϵ / fixed dim.	$d = 2,\ T > T_c$ quasi-exact	$d = 3,\ T < T_c$ LT/ ϵ / fixed dim.
c_2	$-(3.0 - 7.1)\times10^{-4}$	$-7.936\ldots\times10^{-4}$	$\simeq -10^{-2}$
c_3	$(0.5 - 1.3)\times10^{-5}$	$1.096\ldots\times10^{-5}$	$\simeq 4 \times 10^{-3}$
c_4	$-(0.3 - 0.6)\times10^{-6}$	$-0.3127\ldots\times10^{-6}$	
c_5		$0.1267\ldots\times10^{-7}$	
c_6		$-0.6300\ldots\times10^{-9}$	

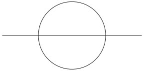
Moreover, the DE

$$\frac{\Gamma^{(2)}(p)}{\Gamma^{(2)}(0)} = 1 + \frac{p^2}{m^2} + \sum_{n=2}^{\infty} c_n \left(\frac{p^2}{m^2}\right)^n$$

has a finite radius of convergence given by the location of the first pole in the complex p^2 plane: It is at the first multiparticle threshold.

Two possibilities:

- in the symmetric phase, it occurs at $p^2 = 9m^2$ because $3m$ is the energy of the lightest 3-particle state:

at one loop $\rightarrow$ 

Thus, $|c_{n+1}/c_n| = 1/9$ when $n \rightarrow \infty$.

- In the broken phase, it occurs at $p^2 = 4m^2$ because there is a tri-linear coupling and thus the threshold corresponds to the creation of 2 particles. Thus, $|c_{n+1}/c_n| = 1/4$ when $n \rightarrow \infty$.

Actually, even in the symmetric phase there are singularities in the 4-point function at $p^2 = 4m^2 \Rightarrow$ the $1/4$ (and not $1/9$) expansion parameter is largely model-independent!

Back to the DE in NPRG

Problem: We are interested in the DE in the presence of $R_k(q)$ at criticality (the worst case) and not away from criticality at $R_k = 0$.

However, R_k plays the role of a (complicated) mass term: it drives a critical system away from criticality and (among other things) generates a mass $m_k \sim k$.
$\Rightarrow$ The DE should also have a finite radius of convergence $\mathcal{R}_c$ and we expect $4 \leq \mathcal{R}_c \leq 9$.
$\Rightarrow$ $\mathcal{R}_c >$ effective range of the integral over q in the flow equations which is $\simeq k$, at least for a regulator that decays rapidly for $q^2 > k^2$.
$\Rightarrow$ It is legitimate to replace any $\Gamma_k^{(n)}$ by its DE inside a flow equation and for vanishing external momenta.

We can notice that $1 \gg |c_2| \gg |c_3| \cdots \gg |c_6|$.
Can we qualitatively explain this behavior?

- $|c_{n+1}| \simeq |c_n|/9 \Rightarrow$ neglect c_3 and beyond. For $p < \mathcal{R}_c$:

$$\frac{\Gamma_k^{(2)}(p)}{\Gamma_k^{(2)}(0)} \simeq 1 + \frac{p^2}{k^2} + \sum_{n=2}^{\infty} c_n \left(\frac{p^2}{k^2}\right)^n \approx 1 + \frac{p^2}{k^2} + c_2 \left(\frac{p^2}{k^2}\right)^2$$

- On the other hand, for $p \gg k$, $\frac{\Gamma_k^{(2)}(p)}{\Gamma_k^{(2)}(0)} \sim (p/k)^{2-\eta}$.
- For $p \sim k$, let us make the guess ($\eta \ll 1$!)

$$\frac{\Gamma_k^{(2)}(p)}{\Gamma_k^{(2)}(0)} \sim 1 + (p/k)^{2-\eta} \approx 1 + \frac{p^2}{k^2} - \eta \frac{p^2}{k^2} \log(p/k)$$

- Matching both expressions at $p^2 = 9k^2$, yields:

$$c_2 \approx -\frac{\eta}{9} \approx -4 \times 10^{-3}$$

- This matches qualitatively with something intermediate between low and high temperature phases. It also explains why the series of the c_n is alternate.

The DE at order 6 for the Ising model in $d = 3$

- The ansatz is

$$\begin{aligned}\Gamma_k[\phi] \; = & \int d^d x \Big[U_k(\phi) + \tfrac{1}{2} Z_k(\phi)(\partial\phi)^2 \\ & + \tfrac{1}{2} W_k^a(\phi)(\partial_\mu\partial_\nu\phi)^2 + \tfrac{1}{2}\phi W_k^b(\phi)(\partial^2\phi)(\partial\phi)^2 \\ & + \tfrac{1}{2} W_k^c(\phi)\left((\partial\phi)^2\right)^2 + \tfrac{1}{2}\tilde{X}_k^a(\phi)(\partial_\mu\partial_\nu\partial_\rho\phi)^2 \\ & + \tfrac{1}{2}\phi\tilde{X}_k^b(\phi)(\partial_\mu\partial_\nu\phi)(\partial_\nu\partial_\rho\phi)(\partial_\mu\partial_\rho\phi) \\ & + \tfrac{1}{2}\phi\tilde{X}_k^c(\phi)\left(\partial^2\phi\right)^3 + \tfrac{1}{2}\tilde{X}_k^d(\phi)\left(\partial^2\phi\right)^2(\partial\phi)^2 \\ & + \tfrac{1}{2}\tilde{X}_k^e(\phi)(\partial\phi)^2(\partial_\mu\phi)(\partial^2\partial_\mu\phi) + \tfrac{1}{2}\tilde{X}_k^f(\phi)(\partial\phi)^2(\partial_\mu\partial_\nu\phi)^2 \\ & + \tfrac{1}{2}\phi\tilde{X}_k^g(\phi)\left(\partial^2\phi\right)\left((\partial\phi)^2\right)^2 + \tfrac{1}{96}\tilde{X}_k^h(\phi)\left((\partial\phi)^2\right)^3 \Big]. \end{aligned} \qquad (1)$$

- We have implemented the "true" DE, that is, we eliminate in all flows all the terms in momenta higher than 6.

Quantities that have been computed and choice of regulator

Check our results with very accurate results in the literature $\Rightarrow$ computation of critical exponents: numerically exact results from the conformal bootstrap.

A major difficulty: all results should be independent of the choice of regulator but a spurious dependence appears at any finite order of the DE.

Does this dependence decrease with the order of the DE?

NO

$\Rightarrow$ An optimisation procedure must be used to reduce the regulator-dependence: this is the "Principle of minimal sensitivity".

$$
\begin{aligned}
W_k(q^2) &= \alpha Z_k^0 k^2\, y/(\exp(y) - 1) \\
\Theta_k^n(q^2) &= \alpha Z_k^0 k^2\, (1-y)^n \theta(1-y) \quad n \in \mathbb{N} \\
E_k(q^2) &= \alpha Z_k^0 k^2\, \exp(-y)
\end{aligned}
$$

with $y = q^2/k^2$ and Z_k^0 an appropriate running renormalization factor.

Results of the DE at order 6 for the Ising model

- α is optimized according to the Principle of Minimal Sensitivity.
- After optimization, the results are very similar for different families of $R_k(q)$.
- For example, exponents for the exponential regulator:

D.E.	ν	$\lvert\delta\nu\rvert$	η	$\lvert\delta\eta\rvert$
$s=0$	0.65103	0.02106	0	0.03630
$s=2$	0.62752	0.00245	0.04551	0.00921
$s=4$	0.63057	0.00060	0.03357	0.00273
$s=6$	0.63007	0.00010	0.03648	0.00018
Conf. Boot.[1]	0.629971(4)		0.0362978(20)	
6-loop [2]	0.6304(13)		0.0335(25)	
High-T. [3]	0.63012(16)		0.03639(15)	
M.-C. [4]	0.63002(10)		0.03627(10)	

[1] Kos et al., JHEP **1411** (2014). [2] Guida, Zinn-Justin, J.Phys. A31 (1998).
[3] Campostrini et al., PRB65 (2002). [4] Hasenbusch, PRB82 (2010).

Taking into account:

1. The variations of the optimized results between families of regulators (which are very small),

2. The convergence of these optimized results with a decrease of their distance to the "exact" results by a factor between 4 and 9,

we can predict the values of the exponents together with error bars (for the first time): $\nu = 0.6300(2)$ and $\eta = 0.0358(6)$ to be compared with

Conf. Boot.[1]	0.629971(4)	0.0362978(20)
6-loop [2]	0.6304(13)	0.0335(25)
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Back to the DE

- Let us now evaluate the behavior of the DE from the results at DE6.
- In the NPRG, the expansion for $\Gamma_k^{(2)}(p,\phi)+R_k(0)$ is:

$$\frac{\Gamma_k^{(2)}(p,\phi)+R_k(0)}{\Gamma_k^{(2)}(0,\phi)+R_k(0)} = 1+\frac{Z_k p^2+W_k^a p^4+X_k^a p^6}{U_k''+R_k(0)}$$

$$\xrightarrow[k\to 0]{} 1+\frac{p^2}{m_{\rm eff}^2}+\frac{w_a^* v^{*\prime\prime}}{z^{*2}}\frac{p^4}{m_{\rm eff}^4}+\frac{x_a^* v^{*\prime\prime 2}}{z^{*3}}\frac{p^6}{m_{\rm eff}^6}$$

- Here u^*, z^*, w_a^*, x_a^* are the dimensionless functions of $\tilde{\phi}$ at the fixed point.
- $m_{\rm eff}^2 = k^2 v^{*\prime\prime}/z^*$ with $v^{*\prime\prime} = u^{*\prime\prime} + R_k(0)/Z_k^0 k^2$.
- The coefficients are now functions of the dimensionless field $\tilde{\phi}$.

- Let us now compare successive orders with different R_k:

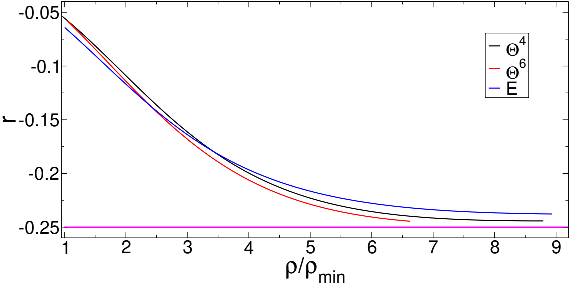

Figure: The ratio $r = x_a^* u^{*\prime\prime}/(w_a^* z^*)$ as a function of $\tilde{\rho} = \tilde{\phi}^2/2$. The line $r = 0.25$ is a guide for the eyes.

- At large fields the successive orders seem to be dominated by the $p^2 = -4m_{eff}^2$ pole.
- At lower fields the ratio r satisfies $|r| < 1/4$.

Summary for Ising in $d = 3$

- At large orders, the DE has a small expansion parameter of order $1/4 \to 1/9$ for model-independent reasons! (but this is not so for Wilson-Polchinski).
- The series for correlation functions alternates (at least at large orders).
- All but leading coefficients are suppressed but an η factor!
- In the NPRG context we need:
  - A regulator that suppresses efficiently momenta $q \gtrsim k$.
  - A regulator that does not introduce singularities in the complex plane for $|q^2| < 4k^2$.
  - The quality of leading orders depends on the value of η.

A side remark: not two independent arbitrariness (the prefactor α of the regulator and the renormalization point where η is computed) by only one.

What about the O(N) models?

Very recent calculations of critical exponents for O(N) at order $s = 4$ in $d = 3$ by G. De Polsi, I. Balog, M. Tissier and N. Wschebor.

All results above are confirmed: convergence with a factor between $1/4$ and $1/9$, efficiency of the PMS to select optimal regulators, very good quality of the results and small error bars (sometimes the best known results).

$\Rightarrow$ The conclusions drawn above seem robust!

What about the PMS?

The DE does not converge for generic regulators: It is only after optimization that it does.
In fact, the dependence of physical results on the regulator increases with the order of the DE.
$\Rightarrow$ the PMS plays a crucial role whereas it is the least understood aspect of our study.
Recent progress has been made (unpublished) by De Polsi, Tissier and Wschebor: the conformal invariance is broken by the DE and minimizing the breaking of this symmetry selects regulators that are (almost) identical to those selected by the PMS! More soon...