## Derivation of the ERGE

These notes are written for a single Euclidean scalar field $\chi$ but can be generalized to multiplets of fields, and to theories with gauge invariances.

In the functional integral approach to quantization we are instructed to calculate the generating functional of connected Green functions $W[J]$, defined by

$$e^{-W[J]} = \int (D\chi) e^{-(S[\chi]+\int J\chi)}$$

and the Legendre transform of $W$:

$$\Gamma[\phi] = W[J] - \int J\phi$$

where $\phi = \langle\chi\rangle = \frac{\delta W}{\delta J}$, which is the generator of 1PI Green functions. In these generating functionals all modes of the quantum field $\chi$ are integrated out. On the other hand when we perform experiments at some energy scale $k$, the effective theory describing the dynamics at that scale is obtained by integrating out all fluctuations of the fields with momenta $p > k$. Wilson described a functional, the effective action $S_k$, which is obtained by cutting off the functional integral at momentum $k$, and described its flow with $k$. We will proceed here much in the same spirit, but with two important differences. First, we are interested in a $k$-dependent analog of $\Gamma$, so the definition will involve a Legendre transform. Second, we will not introduce a sharp cutoff in the functional integral but rather a smooth one. More precisely, instead of cutting off the integral over the low momentum modes of $\chi$, we will introduce a suppression factor for the contribution of such modes. When the suppression becomes infinitely strong, the effect is the same as not integrating on such modes at all.

To implement the above ideas, let us define a $k$-dependent generating functional $W_k[J]$ by

$$e^{-W_k[J]} = \int (D\chi) e^{-(S[\chi]+\Delta S_k[\chi]+\int J\chi)} \qquad (1)$$

Here

$$\Delta S_k[\chi] = \frac{1}{2}\int d^4q\, \chi(-q)\mathcal{R}_k(q^2)\chi(q) \qquad (2)$$

is a new term quadratic in the fields, which is meant to suppress the contribution of modes with $q < k$. This is obtained by demanding $\mathcal{R}_k(q^2)$ to be a cutoff function, by which we mean that it is a monotonically decreasing function of $q^2$ and a monotonically increasing function of $k$, that $\mathcal{R}_k(q^2) \to 0$ for $q \to \infty$ faster than any polynomial, and $\mathcal{R}_k(q^2) \to k^2$ for $q \to 0$. Aside from these requirements, the form of the cutoff is completely arbitrary.

We then define the Legendre transform of $W_k$ by

$$\tilde{\Gamma}_k[\phi] = W_k[J] - \int J\phi \ , \qquad (3)$$

where

$$\phi = \frac{\delta W_k}{\delta J} \qquad (4)$$

and $J$ in the r.h.s. of (3) is to regarded as a function of $\phi$, obtained by inverting (4). Finally we define the average effective action $\Gamma_k$ by subtracting from $\tilde{\Gamma}_k$ the suppression factor that was introduced in the beginning:

$$\Gamma_k[\phi] = \tilde{\Gamma}_k[\phi] - \Delta S_k[\phi] \ . \tag{5}$$

We emphasize at this point that the vertices have not been modified. The only effect of $\Delta S_k$ is to replace the free massles propagator $q^2$ by the cutoff propagator $\mathcal{P}_k(q^2) = q^2 + \mathcal{R}_k(q^2)$. Also note that this is an *infrared* cutoff: its effect is to give a mass of order $k$ to the modes with $p < k$, and no mass to the modes with $p > k$, so it introduces a mass gap in the excitation spectrum of the field. However, curing IR divergences is not its primary purpose: rather, it is a way of introducing explicitly a $k$ dependence in the functional integral. We also notice that when $k \to 0$, $\Delta S_k \to 0$ and therefore $\Gamma_k$ reduces to the ordinary effective action $\Gamma$, where all fluctuations have been integrated out unsuppressed.

In order to begin understanding the motivation behind the definition, we can try to evaluate this functional at one loop. Recall that the ordinary one loop effective action is given by

$$\Gamma^{(1)} = S + \frac{1}{2}\mathrm{Tr}\log\frac{\delta^2 S}{\delta\chi\delta\chi}$$

By the same calculation, the one loop average effective action is

$$\Gamma^{(1)}_k = S + \Delta S_k + \frac{1}{2}\mathrm{Tr}\log\frac{\delta^2(S+\Delta S_k)}{\delta\chi\delta\chi} - \Delta S_k = S + \frac{1}{2}\mathrm{Tr}\log\left(\frac{\delta^2 S}{\delta\chi\delta\chi} + \mathcal{R}_k\right)$$

Note that the term $\Delta S_k$ has canceled out in the r.h.s, so that the only modification has been the replacement of the bare inverse propagator by the cutoff inverse propagator. This provides some rationale for the definition (5).

If we now take the derivative with respect to $t = \log k$ we obtain the following equation:

$$\frac{d\Gamma^{(1)}_k}{dt} = \frac{1}{2}\mathrm{Tr}\left(\frac{\delta^2 S}{\delta\chi\delta\chi} + \mathcal{R}_k\right)^{-1}\frac{d\mathcal{R}_k}{dt} \tag{6}$$

This equation contains all the one loop beta functions of the theory. If we expand

$$\Gamma_k(\chi) = \sum_i g_i(k)\mathcal{O}_i(\chi)$$

where $\mathcal{O}_i[\chi]$ are functional constructed with the field and its derivatives, and $g_i(k)$ are running couplings, taking derivative we have

$$\frac{d\Gamma_k}{dt} = \sum_i \frac{dg_i}{dt}\mathcal{O}_i = \sum_i \beta_{g_i}\mathcal{O}_i$$

and so comparing with the r.h.s. of (6) one can in principle read off all the one loop beta functions.

In the r.h.s. of (6) there appears the bare cutoff inverse propagator. One may guess that the RG improvement of this equation, namely the equation obtained by replacing $S$ by $\Gamma_k$ in the r.h.s., gives a more accurate description of physics. We shall now show that this improved equation is actually *exact*.

Let us begin by deriving the functional $W_k$:

$$\frac{dW_k}{dt} = \frac{d}{dt}\langle \Delta S_k \rangle = \frac{1}{2}\mathrm{Tr}\langle \chi\chi \rangle \frac{d\mathcal{R}_k}{dt}$$

where the trace is an integration over coordinate and momentum space. Then using (4)

$$\begin{aligned}\frac{d\Gamma_k[\phi]}{dt} &= \frac{dW_k}{dt} - \frac{d\Delta S_k[\phi]}{dt} = \\ &= \frac{1}{2}\mathrm{Tr}(\langle\chi\chi\rangle - \langle\chi\rangle\langle\chi\rangle)\frac{d\mathcal{R}_k}{dt} \\ &= -\frac{1}{2}\mathrm{Tr}\frac{\delta^2 W_k}{\delta J \delta J}\frac{d\mathcal{R}_k}{dt}\end{aligned}$$

Next recall the identity

$$\frac{\delta^2 W_k}{\delta J \delta J} = -\left(\frac{\delta^2 \tilde{\Gamma}_k}{\delta\phi\delta\phi}\right)^{-1}$$

which follows from (4) and

$$\frac{\delta\tilde{\Gamma}_k}{\delta\phi} = -J\ .$$

Using this identity and (5) we arrive at the equation

$$\frac{d\Gamma_k}{dt} = \frac{1}{2}\mathrm{Tr}\left(\frac{\delta^2\Gamma_k}{\delta\phi\delta\phi} + \mathcal{R}_k\right)^{-1}\frac{d\mathcal{R}_k}{dt} \tag{7}$$

As mentioned above, this equation is identical to the one loop equation except for the replacement of the bare action by $\Gamma_k$ in the r.h.s.. In fact, the equation does not contain any reference to the bare action anymore, it is written entirely in terms of $\Gamma_k$. Furthermore, the trace in the r.h.s. is finite due to the fast decreasing properties of the cutoff for $p > k$, so that no UV regulator is needed in defining the r.h.s.. This means that the equation contains no reference to UV physics, its is written entirely in terms of renormalized quantities. So, irrespective of the renormalizability or lack thereof of a theory, if one is able to expand the trace on the r.h.s. on the basis of operators $\mathcal{O}_i$ then, as indicated above, one can compute all the beta functions of the theory.

Finally we observe that one can use this equation to study the UV limit of a theory. The ERGE (7) was derived formally from a functional integral, which is an ill defined quantity. However, as mentioned above, the ERGE itself is perfectly well defined. So one can assume that a theory is described by some functional $\Gamma_k$ belonging to some properly defined functional space, and impose that it obeys (7). The flow of this functional for $k \to \infty$ tells us directly whether on a given trajectory the renormalized (physical) quantities diverge or not.