One loop flow in perturbative gravity

The simplest gravitational Lagrangian is of the form

$$V - 2ZR\ . \tag{1}$$

Here $Z = \frac{1}{16\pi G}$, where G is Newton's constant, can be regarded as the wave function renormalization of the graviton and $V = 2Z\Lambda$, where Λ is the cosmological constant, can be seen as the vacuum energy.

Dimensional analysis suggests (and calculations confirm) that the one-loop beta functions of V and Z are

$$\begin{aligned}\frac{dV}{dt} =& ak^4\ ,\\ \frac{dZ}{dt} =& bk^2\ ,\end{aligned} \tag{2}$$

with a and b dimensionless constants.

In terms of the dimensionless variables $\tilde{V} = Vk^{-4}$ and $\tilde{Z} = Zk^{-2}$

$$\begin{aligned}\frac{d\tilde{V}}{dt} =& -4\tilde{V} + a\ ,\\ \frac{d\tilde{Z}}{dt} =& -2\tilde{Z} + b\ ,\end{aligned} \tag{3}$$

This system of equations has a FP at

$$\tilde{V}_* = a/4\ ; \qquad \tilde{Z}_* = b/2. \tag{4}$$

The linearized flow at this FP is described by the matrix

$$M = \begin{pmatrix} -4 & 0 \\ 0 & -2 \end{pmatrix} \tag{5}$$

whose eigenvalues are just the canonical dimensions of V and Z. So the FP is attractive in both directions.

If we rewrite the equations for the more familiar variables G and Λ they have the form

$$\begin{aligned}\frac{d\tilde{\Lambda}}{dt} =& -2\tilde{\Lambda} + 8\pi a\tilde{G} - 16\pi b\tilde{G}\tilde{\Lambda}\ ,\\ \frac{d\tilde{G}}{dt} =& 2\tilde{G} - 16\pi b\tilde{G}^2\ ,\end{aligned} \tag{6}$$

This system of equations has two FP's: a Gaussian FP at $\tilde{G} = 0$, $\tilde{\Lambda} = 0$, and a nontrivial FP at $\tilde{G} = 1/(8\pi b)$ and $\tilde{\Lambda} = a/(4b)$. The latter corresponds to (4), the former cannot be seen in terms of Z.

The linearized flow in terms of these variables is given by

$$M' = \begin{pmatrix} \frac{\partial \tilde{\beta}_{\tilde{\Lambda}}}{\partial \tilde{\Lambda}} & \frac{\partial \tilde{\beta}_{\tilde{\Lambda}}}{\partial \tilde{G}} \\ \frac{\partial \tilde{\beta}_{\tilde{G}}}{\partial \tilde{\Lambda}} & \frac{\partial \tilde{\beta}_{\tilde{G}}}{\partial \tilde{G}} \end{pmatrix} = \begin{pmatrix} -2 - 16\pi b\tilde{G} & 8\pi a - 16\pi b\tilde{\Lambda} \\ 0 & 2 - 32\pi b\tilde{G} \end{pmatrix} \tag{7}$$

At the Gaussian FP this matrix becomes

$$M = \begin{pmatrix} -2 & 8\pi a \\ 0 & 2 \end{pmatrix} \tag{8}$$

which has the canonical dimensions of Λ and G on the diagonal, as expected.

At the nontrivial FP it becomes

$$M = \begin{pmatrix} -4 & 4\pi a \\ 0 & -2 \end{pmatrix} \tag{9}$$

The eigenvalues are the same as for the matrix (5). This was to be expected: according to the general theorem discussed before, the critical exponents do not change under regular coordinate transformations in the space of the couplings. The relation between (Λ, G) and (V, Z) is invertible at the nontrivial FP, and therefore the eigenvalues of M are the same. The same relation is singular at the Gaussian FP, so the general theorem does not apply.

In both cases, the eigenvectors do not point along the Λ and G axes. At the Gaussian FP the "attractive" eigenvector is in the direction $(1, 0)$ but the 'repulsive" one is in the direction $(2\pi a, 1)$. The slant is proportional to a and can therefore be seen as a direct consequence of the running of the vacuum energy in (2).