Linearized Renormalization Group Flow

Consider a theory with (possibly infinitely many) couplings g_i. The renormalization group (RG) gives the dependence of the couplings on an energy scale k. We will use the dimensionless RG parameter $t = \log \frac{k}{k_0}$, where k_0 is an arbitrary reference scale. Assume that the beta functions of a theory are known. In general, they will be functions of the couplings and of the independent scale k:

$$\frac{dg_i}{dt} \equiv k \frac{dg_i}{dk} = \beta_i(g_j, k) \ . \tag{1}$$

Define the dimensionless couplings

$$\tilde{g}_i = k^{-d_i} g_i \ , \tag{2}$$

where d_i is the canonical mass dimension of g_i. The coupling $\tilde{g}_i$ is just g_i measured in units of k. From dimensional analysis, $\beta_i(g_j, k) = k^{d_i} \alpha_i(\tilde{g}_j)$ where $\alpha_i(\tilde{g}_j) = \beta_i(g_j k^{-d_j}, 1)$. The beta functions of the dimensionless variables are given by

$$\tilde{\beta}_i(\tilde{g}_j) = \frac{d\tilde{g}_i}{dt} = -d_i \tilde{g}_i + \alpha_i(\tilde{g}_j) \ . \tag{3}$$

They depend on k only implicitly via the $\tilde{g}_j(t)$.

Suppose $\tilde{g}_*$ is a fixed point (FP) of the flow. We can determine the tangent space to the UV critical surface at the FP by linearizing the flow around the FP. Let $y_i = \tilde{g}_i - \tilde{g}_{i*}$ be new coordinates centered at the FP. The linearized flow equations are

$$\partial_t y_i = M_{ij} y_j \ , \tag{4}$$

where

$$M_{ij} = \frac{\partial \tilde{\beta}_i}{\partial \tilde{g}_j}\Big|_* \ . \tag{5}$$

Let S be the linear transformation that diagonalizes M:

$$S^{-1}_{ik} M_{k\ell} S_{\ell n} = \delta_{in} \lambda_n \ ,$$

where λ_n are the eigenvalues of M. The linearized RG equation for the variables

$$z_i = S^{-1}_{ik} y_k$$

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$$\partial_t z_i = \lambda_i z_i \ , \tag{6}$$

so

$$z_i(t) = \exp(\lambda_i t) = \left(\frac{k}{k_0}\right)^{\lambda_i} \ . \tag{7}$$

The coordinates z_i that correspond to positive eigenvalues are repelled by the FP (for growing t), those corresponding to negative eigenvalues are attracted towards it. Therefore, the tangent space to the critical surface at the FP is the space spanned by the eigenvectors with negative eigenvalue. In particular, the dimension of the critical surface is equal to the number of negative eigenvalues of the matrix M.

The Gaussian Fixed Point

A free theory has vanishing beta functions and therefore must be a FP of the RG flow. It is called the Gaussian FP. In the neighborhood of the Gaussian FP one can use perturbation theory. The first term in the r.h.s. of (3) is the classical term due to the canonical dimension of the coupling. The second term is due to quantum corrections. Let us expand this term in Taylor series around the Gaussian FP:

$$\alpha_i(\tilde{g}_j) = a_{ij}\tilde{g}_j + a_{ijk}\tilde{g}_j\tilde{g}_k + \dots \ . \tag{8}$$

Note that there cannot be a constant term otherwise $\tilde{g}_i = 0$ would not be a FP. When used in equation (5) we find

$$M_{ij} = -d_i\delta_{ij} + a_{ij} \ . \tag{9}$$

It is often the case that the expansion (8) begins with the quadratic terms; then the matrix M_{ij} is diagonal with eigenvalues $-d_i$. Even if this does not happen, the matrix a_{ij} only has entries for $j > i$, so the eigenvalues of M_{ij} are again equal to $-d_i$.

There follows that the UV–attractive (relevant) couplings are those that have positive mass dimension. So, near the origin, the UV critical surface is simply the space spanned by the (power counting) renormalizable couplings. In this way we see that asymptotic safety at a Gaussian FP is equivalent to the statement that the theory is perturbatively renormalizable and asymptotically free.

In the theory of critical phenomena the (opposite of the) eigenvalues λ_i are called "critical exponents". For example in a system with a fixed point with a single negative eigenvalue λ the correlation length behaves near the critical temperature T_c as

$$\xi \approx (T - T_c)^{-\nu}$$

where $\nu = -1/\lambda > 0$ is called the "mass critical exponent". We have seen that at a Gaussian FP the critical exponents are equal to the canonical dimensions. More generally, the relevant couplings are those that have positive critical exponent.

Invariance under reparametrizations

Suppose we change the definition of the couplings g_i:

$$g'_i = g'_i(g_j) \ . \tag{10}$$

We can regard this as a coordinate transformation in the space of all couplings. The beta functions can be regarded as a vectorfield in this space, and their transformation under (10) is

$$\beta'_i = \frac{\partial g'_i}{\partial g_j}\beta_j \ . \tag{11}$$

Then, the matrix M transforms as

$$M'_{ij} = \frac{\partial \tilde{g}_k}{\partial \tilde{g}'_j}\frac{\partial^2 \tilde{g}'_i}{\partial \tilde{g}_k \partial \tilde{g}_\ell}\beta_\ell + \frac{\partial \tilde{g}'_i}{\partial \tilde{g}_\ell}M_{\ell k}\frac{\partial \tilde{g}_k}{\partial \tilde{g}'_j}$$

At a FP the first term vanishes, so the matrix M transforms as

$$M'_{ij}\Big|_* = \frac{\partial \tilde{g}'_i}{\partial \tilde{g}_\ell}M_{\ell k}\frac{\partial \tilde{g}_k}{\partial \tilde{g}'_j}\Big|_* \ . \tag{12}$$

This implies that the critical exponents are the same independent of the choice of coordinates in the space of couplings. Of course, if a coordinate transformation is singular at the FP this need not be the case. This happens for example if we choose to describe the Gaussian FP with a coordinate $g' = 1/g$.