The beta function of scalar theory.

We apply the Exact Renormalization Group Equation (ERGE)

$$\partial_t \Gamma_k = \frac{1}{2}\mathrm{Tr}\left(\frac{\delta^2\Gamma_k}{\delta\phi\delta\phi} + R_k\right)^{-1} \partial_t R_k \tag{1}$$

to a Euclidean scalar theory in d spacetime dimensions, which can also be viewed as a statistical mechanical model in d space dimensions. The r.h.s. of the equation can be viewed as a "beta functional" of the theory, containing all the beta functions of all couplings. In order to perform an explicit calculation we consider a truncation of Γ_k of the form

$$\Gamma_k(\phi) = \int d^d x \left[\frac{1}{2}(\partial_\mu \phi)^2 + V_k(\phi^2)\right] \ , \tag{2}$$

where V_k is a k-dependent potential. We will insert this ansatz i (1) and extract the beta functions of the potential. The inverse propagator corresponding to (2) is

$$\frac{\delta^2\Gamma_k}{\delta\phi\delta\phi} = -\partial^2 + 2V_k' + 4\phi^2 V_k'' \ , \tag{3}$$

where a prime denotes the derivative with respect to ϕ^2. In the definition of the functional Γ_k we modify the inverse propagator by adding to it the kernel $R_k(-\partial^2)$, which in momentum space is simply a function of q^2. The modified inverse propagator is

$$\frac{\delta^2\Gamma_k}{\delta\phi\delta\phi} + R_k = P_k(-\partial^2) + 2V_k' + 4\phi^2 V_k'' \ , \tag{4}$$

where we have defined the function $P_k(z) = z + R_k(z)$ (for any argument). With these definitions the ERGE becomes

$$\partial_t \Gamma_k = \frac{1}{2}\mathrm{Tr}\left(\frac{\partial_t P_k}{P_k + 2V_k' + 4\phi^2 V_k''}\right) . \tag{5}$$

(We have used $\partial_t R_k = \partial_t P_k$.) The trace involves an integration over spacetime and over momenta. For any function W,

$$\mathrm{Tr}(W(-\partial^2)) = \int d^d x \int \frac{d^d p}{(2\pi)^d} W(q^2) = A_d \int d^d x \, Q_{\frac{d}{2}}(W) \ . \tag{6}$$

In the last step we have performed the angular integration and we have defined

$$Q_n[W] = \frac{1}{\Gamma(n)} \int_0^{+\infty} dz \, z^{n-1} W(z) \ . \tag{7}$$

where W is a function of $z = |p|^2$. Thus, the function Q contains the integration over the modulus of the momentum. The constant A_d is equal to

$$A_d = \frac{1}{2}\frac{1}{(2\pi)^d}\mathrm{Vol}(S^{d-1})\Gamma(d/2) = \frac{1}{(4\pi)^{d/2}} \ ,$$

where $\mathrm{Vol}(S^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the volume of the $(d-1)$-dimensional sphere. If we now restrict ourselves to constant scalar fields, we can remove a volume factor from both sides of the ERGE and we obtain the k-dependence of the potential as:

$$\partial_t V_k = \frac{1}{2} A_d Q_{\frac{d}{2}}\left(\frac{\partial_t P_k}{P_k + 2V_k' + 4\phi^2 V_k''}\right) \ . \tag{8}$$

There are various ways of studying this equation. In order to make connection with familiar formulae let us consider potentials that admit a Taylor series of the form

$$V(\phi^2) = \sum_{n=1}^{N} \lambda_{2n}\phi^{2n} \ . \tag{9}$$

We will use the ERGE to derive the beta functions of the couplings in the potential. The coupling constants can be extracted from the potential by

$$\lambda_{2n} = \frac{1}{n!}\frac{\partial^n V}{\partial(\phi^2)^n}\Big|_{\phi=0} \ . \tag{10}$$

and the beta functions can be extracted from the "beta functional" (8) by

$$\beta_{2n} = \partial_t \lambda_{2n} = \frac{1}{n!}\frac{\partial^n}{\partial(\phi^2)^n}\partial_t V_k\Big|_{\phi=0} \ . \tag{11}$$

Explicitly, the first few beta functions are given by

$$\begin{aligned}
\beta_2 =& \frac{A_d}{2}\left[-12\lambda_4 Q_{\frac{d}{2}}\left(\frac{\partial_t P_k}{(P_k+2\lambda_2)^2}\right)\right] ,\\
\beta_4 =& \frac{A_d}{2}\left[-30\lambda_6 Q_{\frac{d}{2}}\left(\frac{\partial_t P_k}{(P_k+2\lambda_2)^2}\right) + 144\lambda_4^2 Q_{\frac{d}{2}}\left(\frac{\partial_t P_k}{(P_k+2\lambda_2)^3}\right)\right] ,\\
\beta_6 =& \frac{A_d}{2}\left[-56\lambda_8 Q_{\frac{d}{2}}\left(\frac{\partial_t P_k}{(P_k+2\lambda_2)^2}\right) + 720\lambda_4\lambda_6 Q_{\frac{d}{2}}\left(\frac{\partial_t P_k}{(P_k+2\lambda_2)^3}\right) - 1728\lambda_4^3 Q_{\frac{d}{2}}\left(\frac{\partial_t P_k}{(P_k+2\lambda_2)^4}\right)\right] ,\\
\beta_8 =& \frac{A_d}{2}\Bigg[- 90\lambda_{10} Q_{\frac{d}{2}}\left(\frac{\partial_t P_k}{(P_k+2\lambda_2)^2}\right) + 1344\lambda_4\lambda_8 Q_{\frac{d}{2}}\left(\frac{\partial_t P_k}{(P_k+2\lambda_2)^3}\right)\\
&+ 900\lambda_6^2 Q_{\frac{d}{2}}\left(\frac{\partial_t P_k}{(P_k+2\lambda_2)^3}\right) - 12960\lambda_6\lambda_4^2 Q_{\frac{d}{2}}\left(\frac{\partial_t P_k}{(P_k+2\lambda_2)^4}\right) + 20736\lambda_4^4 Q_{\frac{d}{2}}\left(\frac{\partial_t P_k}{(P_k+2\lambda_2)^5}\right)\Bigg] .
\end{aligned} \tag{12}$$

Each term in the r.h.s. can be represented as a one loop diagram with $2n$ external legs.

When looking for fixed points, and more generally when doing numerical work, one has to reduce everything to dimensionless variables. We assume that units are such that $\hbar = 1$, $c = 1$, so that everything has dimension of a power of mass. Then, we take k as unit of mass and we measure everything else in units of k. Thus, we define the dimensionless variables

$$\tilde\lambda_{2n} = k^{(d-2)n-d}\lambda_{2n} \ , \tag{13}$$

which are the couplings measured in units of k. Their beta functions are

$$\partial_t\tilde\lambda_{2n} = ((d-2)n-d)\tilde\lambda_{2n} + k^{(d-2)n-d}\beta_{2n} \ . \tag{14}$$

The Wilson–Fisher fixed point.

We will use these beta functions to derive the Wilson–Fisher fixed point. Let us now restrict ourselves to $d = 3$ and consider the simplest truncation $N = 2$ in (9). Then the beta functions are

$$\begin{aligned}
\partial_t\tilde\lambda_2 =& -2\tilde\lambda_2 - 6A_3\tilde\lambda_4 k^{-1} Q_{\frac{3}{2}}\left(\frac{\partial_t P}{(P+2\lambda_2)^2}\right)\\
\partial_t\tilde\lambda_4 =& -\tilde\lambda_4 + 72A_3\tilde\lambda_4^2 k Q_{\frac{3}{2}}\left(\frac{\partial_t P}{(P+2\lambda_2)^3}\right)
\end{aligned} \tag{15}$$

In order to be able to perform the integrals in Q in closed form we will use the so-called "optimized cutoff function" of Litim (2001)

$$R_k(z) = (k^2 - z)\theta(k^2 - z) \ . \tag{16}$$

With this cutoff $\partial_t R_k = 2k^2\theta(k^2 - z)$. Since the integrals are all cut off at $z = k^2$ by the theta function in the numerator, we can simply use $P_k(z) = k^2$ in the integrals. For $n > 0$ we have

$$Q_n\left(\frac{\partial_t P}{(P+A)^\ell}\right) = \frac{2}{n!}\frac{1}{(1+\tilde{a})^\ell}k^{2(n-\ell+1)} \ . \tag{17}$$

where $\tilde{a} = ak^{-2}$. Then the beta functions become

$$\begin{aligned} \partial_t\tilde{\lambda}_2 &= -\,2\tilde{\lambda}_2 - \frac{1}{\pi^2}\frac{2\tilde{\lambda}_4}{(1+2\tilde{\lambda}_2)^2} \ , \\ \partial_t\tilde{\lambda}_4 &= -\,\tilde{\lambda}_4 + \frac{1}{\pi^2}\frac{24\tilde{\lambda}_4^2}{(1+2\tilde{\lambda}_2)^3} \ . \end{aligned} \tag{18}$$

These beta functions have two simultaneous zeroes. One is in the origin $\tilde{\lambda}_2 = \tilde{\lambda}_4 = 0$ and is called the Gaussian fixed point; the other is at

$$\tilde{\lambda}_2 = -\frac{1}{26} \approx -0.03846 \ ; \quad \tilde{\lambda}_4 = \frac{72\pi^2}{2197} \approx 0.3234 \ . \tag{19}$$

and is called the Wilson-Fisher fixed point.

The attractivity properties of a fixed point can be determined by studying the linearized flow in its neighborhood. The linearized flow equations are

$$\partial_t y_i = M_{ij} y_j \ , \tag{20}$$

where $y_i = \tilde{\lambda}_i - \tilde{\lambda}_{i*}$ and

$$M_{ij} = \frac{\partial\tilde{\beta}_i}{\partial\tilde{g}_j}\Big|_* \ . \tag{21}$$

$$M_* = \begin{pmatrix} \frac{\partial\tilde{\beta}_2}{\partial\tilde{\lambda}_2} & \frac{\partial\tilde{\beta}_2}{\partial\tilde{\lambda}_4} \\ \frac{\partial\tilde{\beta}_4}{\partial\tilde{\lambda}_2} & \frac{\partial\tilde{\beta}_4}{\partial\tilde{\lambda}_4} \end{pmatrix}\Bigg|_* = \begin{pmatrix} -\frac{5}{3} & -\frac{36\pi^2}{169} \\ -\frac{169}{72\pi^2} & 1 \end{pmatrix} \approx \begin{pmatrix} -1.6667 & -2.1024 \\ -0.2378 & 1 \end{pmatrix} \ . \tag{22}$$

The eigenvalues of this matrix are -1.8425 and 1.1759; the flow approaches the fixed point as k^α, where α are the eigenvalues. For this reason the eigenvalues, or rather minus the eigenvalues, $\theta_1 = 1.8425$ and $\theta_2 = -1.1759$ are called the "critical exponents". In particular, when this theory is used in statistical mechanics as a model for phase transitions in three dimensions, the "mass critical exponent" $\nu = -1/\alpha_1$ determines the scaling of the correlation length near the critical temperature

$$\xi \approx |T - T_c|^{-\nu} \ .$$

In the preceding calculation $\nu \approx 0.5427$. This number can be measured experimentally and has been calculated to high precision using several methods (long before the ERGE was even known). Bonanno and Zappalà (2001) Phys.Lett.B504:181-187 (e-Print: hep-th/0010095) give a table comparing the values of the critical exponents of this theory from various approximations and from experiment. The best values for ν are in the range $0.62 - 0.63$.

One way to improve the approximation is to take into account the effect of higher couplings. This means truncating the potential by a polynomial of sixth or higher order. Instead of proceeding as before, we shall write directly the beta functional for the dimensionless potential. We return to general dimension d. We define the dimensionless field $\tilde{\phi} = k^{\frac{2-d}{2}}\phi$ and the dimensionless potential

$$\tilde{V}(\tilde{\phi}^2) = k^{-d}V_k(\phi^2) = k^{-d}V_k(k^{d-2}\tilde{\phi}^2) \ . \tag{23}$$

Then we have

$$\partial_t \tilde{V} = -d\tilde{V} + (d-2)\tilde{\phi}^2\tilde{V}' + k^{-d}\partial_t V_k \ . \tag{24}$$

Let us now choose the optimized cutoff (16). In this case the momentum integral in (8) can be performed explicitly yielding

$$\partial_t V_k = C_d \frac{k^d}{1 + 2\tilde{V}' + 4\tilde{\phi}^2\tilde{V}''} \ . \tag{25}$$

where

$$C_d = \frac{1}{2}\frac{4}{d}\frac{1}{\Gamma(d/2)}A_d \ . \tag{26}$$

Therefore

$$\partial_t \tilde{V} = -d\tilde{V} + (d-2)\tilde{\phi}^2\tilde{V}' + C_d \frac{1}{1 + 2\tilde{V}' + 4\tilde{\phi}^2\tilde{V}''} \ , \tag{27}$$

For the potential (9), we have

$$\tilde{V}(\tilde{\phi}^2) = \sum_{n=1}^{N} \tilde{\lambda}_{2n}\tilde{\phi}^{2n} \ . \tag{28}$$

The beta functions of $\tilde{\lambda}_{2n}$ can be obtained directly from (27):

$$\partial_t \tilde{\lambda}_{2n} = \frac{1}{n!}\frac{\partial^n}{\partial(\tilde{\phi}^2)^n}\partial_t \tilde{V}_k \Big|_{\tilde{\phi}=0} \ . \tag{29}$$

The attached Mathematica notebook calculates the beta functions and the critical exponents for various truncations. The program is to be used as follows. Set $d = 3$ (or any other dimension, if one is interested in computing the scalar beta functions in other dimensions). Set n to the desired value, e.g. $n = 2$ corresponds to the calculation that was done above. The symbol ρ is used for $\tilde{\phi}^2$. (In general, all tildas are dropped for notational simplicity.) Evaluate lines down to line 13. Output line 13 gives the position of the zeroes of the beta functions. Most of these fixed points are fictitious: they come from having approximated the potential by a polynomial. In line 14, manually select the physical fixed point. For example, `Extract[fp,[3]]` will select the third solution. The choice is not obvious at first. One can immediately discard the solutions with a nonvanishing imaginary part, but there are generally several real solutions. One procedure is to note the position of the fixed point at a given n and then in the truncation at order $n + 1$ identify the solution for which the couplings $\tilde{\lambda}_2 \ldots \tilde{\lambda}_{2n}$ are closest to those of the lower truncation. Pay attention: sometimes all solutions seem to have an imaginary part but it is zero within numerical errors. Having chosen the quantity "wf", the remaining lines extract the matrix M and the critical exponents.

This calculation serves as an introduction to the use of truncations of the ERGE. However, truncating on the field (i.e. approximating the potential by a polynomial) is not the best way of using the ERGE. In the present problem, it is better to treat (27) as a beta functional for the dimensionless potential and look for a function $\tilde{V}_*$ that is a zero of this functional. When possible it is better to use an expansion in derivatives, where one keeps all terms in the action that contain up to a certain number of derivatives. See for example C. Bagnuls and C. Bervilliers (2001).